An overview of factor models for pricing CDO tranches

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Abstract

We review the pricing of synthetic CDO tranches from the point of view of factor models. Thanks to the factor framework, we can handle a wide range of well-know pricing models. This includes pricing approaches based on copulas, but also structural, multivariate Poisson and affine intensity models. Factor models have become increasingly popular since there are associated with efficient semi-analytical methods and parsimonious parameterization. Moreover, the approach is not restrictive at all in the case of homogeneous credit portfolios. Easy to compute and to handle large portfolio approximations can be provided. In factor models, the distribution of conditional default probabilities is the key input for the pricing of CDO tranches. These conditional default probabilities are also closely related to the distribution of large portfolios. Therefore, we can compare different factor models by simply comparing the distribution functions of the corresponding conditional default probabilities.

Keywords: CDOs, bottom-up, top-down, large portfolio approximations, de Finetti's theorem, factor copulas, Archimedean copulas, structural models, intensity models, multivariate Poisson models.

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Introduction

When looking at the pricing methodologies for credit derivatives, a striking feature is the profusion of competing approaches, none of them could be seen as an academic and practitioner's standard. This contrasts with equity or interest rate derivatives to set some examples. Despite rather negative appreciation from the academic world, the industry relies on the one factor Gaussian copula for the pricing of CDO tranches, possibly amended with base correlation approaches. Among the usual critics, one can quote the poor dynamics of the credit loss process, of the credit spreads and the disconnection between the pricing and the hedging, while pricing at the cost of the hedge is a cornerstone of modern finance. Given the likelihood of plain static arbitrage opportunities when "massaging" correlations without caution, the variety and complexity of mapping procedures for the pricing of bespoke portfolios, a purist might assert that base correlations are simply a way to express CDO tranche quotes. Even from that minimal view, the computation of base correlations from market quotes is not an easy task due to the amortization scheme of premium legs and the dependence on more or less arbitrary assumptions on recovery rates.

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Unsurprisingly, there are many ways to assess model quality, such as the ability to fit market quotes, tractability, parsimony, hedging efficiency and of course economic relevance and theoretical consistency. One should keep in mind that different models may be suitable for different payoffs. As discussed below, standard CDO tranche premiums only depend upon the marginal distributions of portfolio losses at different dates, and not on the temporal dependence between losses. This may not be the case for more exotic products such as leverage tranches, forward starting CDOs. Therefore, copula models might be well suited for the former plain vanilla products while a direct modelling of the loss process, as in the top-down approach, tackles the latter. Standard tranches on iTraxx or CDX have almost become asset classes on their own. Though the market directly provides their premium at the current date, a modelling of the corresponding dynamics might be required when risk managing non standard tranches. Let us remark that the informational content of standard tranches is not fully satisfactory, especially when considering the pricing of tranchelets corresponding to first losses (e.g. a $[0,1\%]$ tranche) or senior tranches associated with the right tail of the loss distribution. There are also some difficulties when dealing with short maturity tranches. Whatever the chosen approach, a purely numerical smoothing of base correlations or a pricing model based interpolation, there is usually a lot of model risk: models that are properly calibrated to liquid tranches prices may lead to significantly different prices for non standard tranches.

In the remainder of the paper, we will focus on pricing models for typical synthetic CDO tranches, either based on standard indexes or related to bespoke portfolios and we will not further consider products that involve the joint distribution of losses and credit spreads such as options on tranches. We will focus on model based pricing approaches, such that the premium of the tranche can be obtained by equalling the present value of the premium and the default legs of the tranches, computed under a given risk neutral measure. At least, this rules out static arbitrage opportunities, such as negative tranchelet prices. Thus, we will leave aside comparisons between base correlation and model based approaches that might be important in some cases. Though we will discuss the ability of different models to be well calibrated to standard liquid tranches, we will not further consider the various and sometimes rather proprietary mapping methodologies that aim at pricing bespoke CDO tranches given the correlation smiles on standard indexes. Such practical issues are addressed in Gregory and Laurent (2008) and in the references therein.

Fortunately, there remain enough models to leave anyone with an encyclopaedic tendency more than happy. When so many academic approaches contest, there is a need to categorize, which obviously does not mean to write down a catalogue.

Recently, there has been a discussion about the relative merits of bottom-up and top-down approaches. In the actuarial field, these are also labelled as the individual and the collective models. In a bottom-up approach, also known as a name per name approach, one starts from a description of the dynamics (credit spreads, defaults) of the names within a basket, from which the dynamics of the aggregate loss process is derived. Some aggregating procedure involving the modelling of dependence between the default events is required to derive the loss distribution. The bottom-up approach has some clear advantages over the top-down approach, such as the possibility to easily account for name heterogeneity: for instance the trouble with GMAC and the corresponding widening of spreads had a salient impact on CDX equity tranche quotes. It can be easily seen that the heterogeneity of individual default probabilities breaks down the Markov property of the loss process. One needs to know the current structure of the portfolio, for example the proportion of risky names, and not only the current losses to further simulate appropriately further losses. This issue is analogous to the well-known burnout effect in mortgage prepayment modelling. The random thinning approach only provides a partial answer to the heterogeneity issue: names with higher marginal default probabilities actually tend to default first, but the change in the loss intensity does not depend upon the defaulted name, as one should expect. The concept of idiosyncratic gamma which is quite important in the applied risk management of equity tranches is thus difficult to handle in a top-down approach. Also, a number of models belonging to this class do not account for the convergence to zero of the loss intensity as the portfolio is exhausted. This leads to positive, albeit small, probabilities that the loss exceeds the nominal of the portfolio. Another practical and paramount topic is the risk management of CDO tranches at the book level. Since most investment banks deal with numerous credit portfolios, they need to model a number of aggregate loss processes, which obviously are not independent. While such a global risk management approach is amenable to the bottom-up approach, it remains an open issue for its contender.

On the other hand, there are some other major drawbacks when relying on bottom-up approaches. A popular family within the bottom-up approaches, relying on Cox processes bores its own burden. On theoretical grounds, it fails to account for contagion effects, also known as informative defaults: default of one name may be associated with jumps, usually of positive magnitude, of the credit spreads of the surviving names. Though some progress has recently been completed, the numerical implementation, especially with respect to calibration on liquid tranches is cumbersome. In factor copula approaches, the dynamics of the aggregate loss is usually quite poor, with high dependence between losses at different time horizons and even comonotonic losses in the large portfolio approximation. Thus, factor copula approaches fall into disrepute when dealing with some forward starting tranches where the dependence between losses at two different time horizons is a key input.

Nevertheless, the pricing of synthetic CDO tranches only involves marginal distribution of losses and is likely to be better handled in the bottom-up approach. Since this paper is focused on CDO tranches, when discussing pricing issues, we will favour the name per name perspective.

As mentioned above, due to the number of pricing models at hand[3], there is the need of a unifying perspective, especially with respect to the dependence between default dates. In the following, we will privilege a factor approach: default dates will be independent given a low dimensional factor. This framework is not that restrictive since it encompasses factor copulas, but also multivariate Poisson, structural models and some intensity models within the affine class. Moreover, in the homogeneous case, where the names are indistinguishable, on a technical ground this corresponds to

[3] See Duffie and Singleton (2003), Schönbucher (2003), Bielecki and Rutkowski (2004) or Lando (2004) textbooks for a detailed account of the different approaches to credit risk.

the exchangeability assumption, the existence of a single factor is a mere consequence of de Finetti's theorem as explained below. From a theoretical point of view, the key inputs in a single factor model are the distributions of the conditional (on the factor) default probabilities. Given these, one can unambiguously compute CDO tranche premiums in a semi-analytical way. It is also fairly easy to derive large portfolio approximations under which the pricing of CDO tranche premiums reduces to a simple numerical integration. The factor approach also allows some model taxonomy by comparing the conditional default probabilities through the so-called convex order. This yields some useful results on the ordering of tranche premiums. The factor assumption is also almost necessary to deal with large portfolios and avoid over-fitting. As an example, let us consider the Gaussian copula; the number of correlation parameters evolves as n^2, where n is the number of names, without any factor assumption, while it increase linearly in a one factor model.

In section I, we will present some general features of factor models with respect to the pricing of CDO tranches. This includes the derivation of CDO tranche premiums from marginal loss distributions, the computation of loss distributions in factor models, the factor representation associated with de Finetti's theorem for homogeneous portfolios, large portfolio approximations and an introduction to the use of stochastic orders as a way to compare different models. Section II details various factor pricing models, including factor copula models, structural, multivariate Poisson, and Cox process based models. As for the factor copula models, we deal with additive factor copula models and some extensions involving stochastic or local correlation. We also consider Archimedean copulas and eventually "perfect" copulas that are implied from market quotes. Multivariate Poisson models include the so-called common shock models. Examples based on Cox processes are related to affine intensities while structural models are multivariate extensions of the Black and Cox first hitting time of a default barrier.

I Factor models for the pricing of CDO tranches

Factor models have been used for a long time with respect to stock or mutual fund returns. As far as credit risk management is concerned, factor models also appear as an important tool. They underlie the IRB approach in the Basel II regulatory framework: see Crouhy *et al*. (2000), Finger (2001), Gordy (2000, 2003), Wilson (1997a, 1997b) or Frey and McNeil (2003) for some illustrations. The idea of computing loss distributions from the associated characteristic function in factor models can be found in Pykhtin and Dev (2002). The application of such ideas to the pricing of CDOs is discussed in Gregory and Laurent (2003), Andersen, Sidenius and Basu (2003), Hull and White (2004), Andersen and Sidenius (2005a) and Laurent and Gregory (2005). Various discussions and extensions about the factor approach for the pricing of CDO tranches can be found in a number of papers, including, Finger (2005), Burtschell *et al.* (2008).

I.1 Computation of CDO tranche premiums from marginal loss distributions

A synthetic CDO tranche is a structured product based on an underlying portfolio of equally weighted reference entities subject to credit risk[4]. Let us denote by n the number of references in the credit portfolio and by $(\tau_1,\ldots,\tau_n)$ the random vector of default times. If name i defaults, it drives a loss of $M_i = E(1-\delta_i)$ where E denotes the nominal amount (which is usually name independent for a synthetic CDO) and δ_i the recovery rate. M_i is also referred as the loss given default of name i. The key quantity for the pricing of CDO tranches is the cumulative loss $L_t = \sum_{i=1}^{n} M_i D_i$, where $D_i = 1_{\{\tau_i \le t\}}$ is a Bernoulli random variable indicating whether name i defaults before time t. L_t is a pure jump process and follows a discrete distribution at any time *t*.

The cash-flows associated with a synthetic CDO tranche only depend upon the realized path of the cumulative losses on the reference portfolio. Default losses on the credit portfolio are split along some thresholds (attachment and detachment points) and allocated to the various tranches. Let us consider a CDO tranche with attachment point a, detachment point b and maturity T. It is sometimes convenient to see a CDO tranche as a bilateral contract between a protection seller and a protection buyer. We describe below the cash-flows associated with the default payment leg (payments received by the protection buyer) and the premium payment leg (payments received by the protection seller)..

Default payments leg

The protection seller agrees to pay the protection buyer default losses each time they impact the tranche $[a,b]$ of the reference portfolio. More precisely, the cumulative default payment $L_t^{[a,b]}$ on the tranche $[a,b]$ is equal to zero if $L_t \le a$, to $L_t - a$ if $a \le L_t \le b$ and to $b-a$ if $L_t \ge b$. Let us remark that $L_t^{[a,b]}$ has a call spread payoff with respect to L_t (see Figure 1) and can be expressed as $L_t^{[a,b]} = (L_t - a)^+ - (L_t - b)^+$.

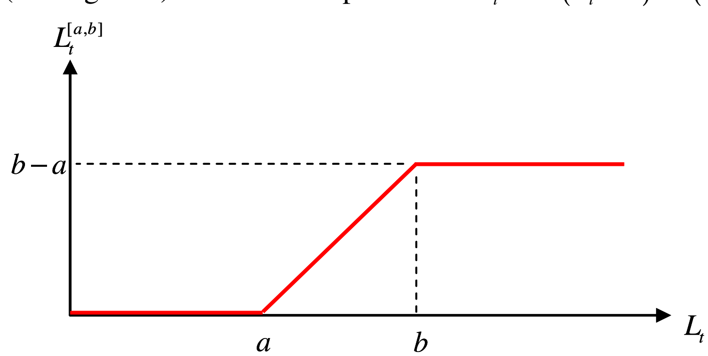

Figure 1. Cumulative loss on CDO tranche $[a,b]$ with respect to L_t

[4] We refer the reader to Das (2005) textbook or Kakodkar *et al.* (2006) for a detailed analysis of the CDO market and credit derivatives cash-flows.

Default payments are simply the increment of $L_t^{[a,b]}$, i.e. there is a payment of $L_t^{[a,b]} - L_{t-}^{[a,b]}$ from the protection seller at every jump time of $L_t^{[a,b]}$ occurring before contract maturity T. Figure 2 shows a realized path of the loss process L_t and consequences on CDO tranche $[a,b]$ cumulative losses.

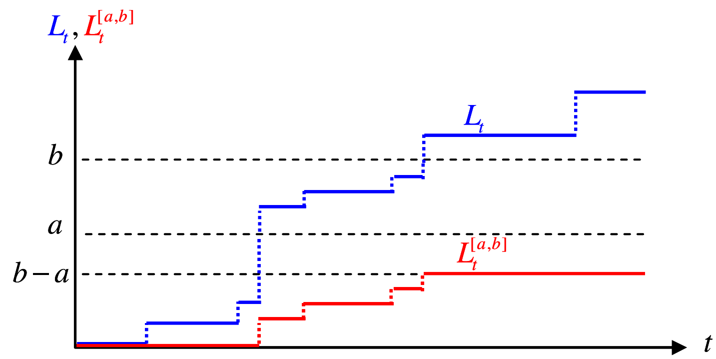

Figure 2. A realized path of the reference portfolio losses in blue and the corresponding path of losses affecting CDO tranche $[a,b]$ in red. Jumps occur at default times.

For simplicity we further assume that the continuously compounded default free interest rate r_t is deterministic and denote by $B_t = \exp\left(-\int_0^t r_s ds\right)$ the discount factor. Then, the discounted payoff corresponding to default payments can written as:

$$\int_0^T B_t dL_t^{[a,b]} = \sum_{i=1}^n B_{\tau_i}\left(L_{\tau_i}^{[a,b]} - L_{\tau_i -}^{[a,b]}\right) 1_{\{\tau_i \le T\}}.$$

Thanks to Stieltjes integration by parts formula and Fubini theorem, the price of the default payment leg can be expressed as:

$$E\left[\int_0^T B_t dL_t^{[a,b]}\right] = B_T E\left[L_T^{[a,b]}\right] - \int_0^T r_t B_t E\left[L_t^{[a,b]}\right] dt.$$

Premium payments leg

The protection buyer has to pay the protection seller a periodic premium payment (quarterly for standardized indexes) based on a fixed spread or premium S and proportional to the current outstanding nominal of the tranche $b-a-L_t^{[a,b]}$. Let us denote by t_i, $i=1,\dots,I$ the premium payment dates with $t_I = T$ and by Δ_i the length of the i^{th} period $\left[t_{i-1}, t_i\right]$ (in fractions of a year and with $t_0 = 0$). The CDO premium payments are equal to $S\Delta_i\left(b-a-L_{t_i}^{[a,b]}\right)$ at regular payment dates t_i, $i=1,\dots,I$. Moreover, when a default occurs between two premium payment dates and when it affects the tranche, an additional payment (the accrued coupon) must be made at default time to compensate the change in value of the tranche outstanding nominal. For example, if name j defaults between t_{i-1} and t_i, the associated accrued coupon is equal to $S\left(\tau_j - t_{i-1}\right)\left(L^{[a,b]}_{\tau_j} - L^{[a,b]}_{\tau_j -}\right)$. Eventually, the discounted payoff corresponding to premium payments can be expressed as:

$$\sum_{i=1}^{I}\left(B_{t_i} S \Delta_i \left(b - a - L^{[a,b]}_{t_i}\right) + \int_{t_{i-1}}^{t_i} B_t S\left(t - t_{i-1}\right) dL^{[a,b]}_t \right).$$

Using same computational methods as for the default leg, it is possible to derive the price of the premium payment leg, that is

$$S\sum_{i=1}^{I}\left(B_{t_i}\Delta_i\left(b - a - E\left[L^{[a,b]}_{t_i}\right]\right) + B_{t_i}\left(t_i - t_{i-1}\right)E\left[L^{[a,b]}_{t_i}\right] - \int_{t_{i-1}}^{t_i} B_t\left(r_t\left(t - t_{i-1}\right) + 1\right)E\left[L^{[a,b]}_t\right]dt \right).$$

The CDO tranche premium S is chosen such that the contract is fair at inception, i.e. such that the default payment leg is equal to the premium payment leg. S is quoted in basis point per annum[5]. Figure 3 shows the dynamics of credit spreads on the five year Itraxx index (series 7) between May and November 2007. It is interesting to observe a wide bump corresponding to the summer 2007 crisis.

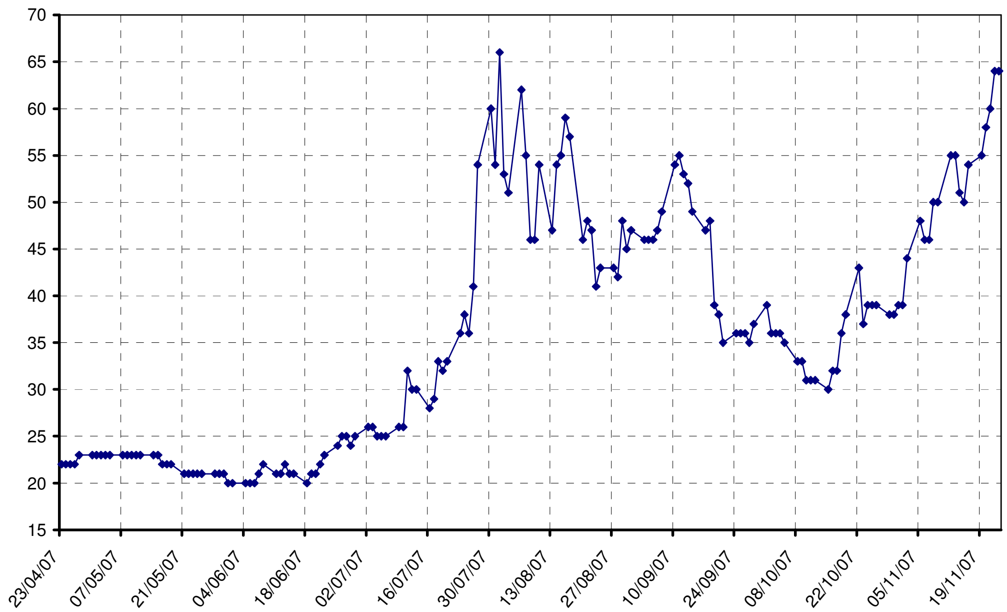

Figure 3. Credit spreads on the five years iTraxx index (Series 7) in bps

Let us remark that the computation of CDO tranche premiums only involves the expected losses on the tranche, $E\left[L^{[a,b]}_t\right]$ at different time horizons. These can readily be derived from the marginal distributions of the aggregate loss on the reference

[5] Let us remark that market conventions are quite different for the pricing of equity tranches (CDO tranches $[0,b]$ with $0 < b < 1$). Due to the high level of risk embedded in these "first losses tranches", the premium S is fixed beforehand at 500 bps per annum and the protection seller receive an additional payment at inception based on an "upfront premium" and proportional to the size b of the tranche. This "upfront premium" is quoted in percentage.

portfolio. In the next section, we describe some numerical methods for the computation of the aggregate loss distribution within factor models.

I.3 Computation of loss distributions

In a factor framework, one can easily derive marginal loss distributions. We will assume that default times are conditionally independent given a one dimensional factor V. The key inputs for the computation of loss distribution are the conditional default probabilities $p_t^{i|V} = P\left(\tau_i \le t \middle| V\right)$ associated with names $i = 1,\dots,n$. Extensions to multiple factors are straightforward but are computationally more involved. However the one factor assumption is not that restrictive as explained in Gössl (2007) where computation of the loss distribution is performed with an admissible loss of accuracy using some dimensional reduction techniques. In some examples detailed below, the factor V may be time dependent. This is of great importance when pricing correlation products that involve the joint distribution of losses at different time horizons such as leverage tranches or forward starting CDOs. Since this paper is focused on the pricing of standard CDO tranches, which only involve marginal distributions of cumulative losses, omitting the time dependence is a matter of notational simplicity.

Unless otherwise stated, we will thereafter assume that recovery rates are deterministic and concentrate upon the dependence among default times.

Two approaches can be used for the computation of loss distributions, one based on the inversion of the characteristic function and another one based on recursions.

FFT approach

The first approach deals with the characteristic function of the aggregate loss L_t which can be derived thanks to the conditional independence assumption:

$$\varphi_{L_t}(u) = E\left[e^{iuL_t}\right] = E\left[\prod_{1 \le i \le n}\left(1 + p_t^{i|V}\left(e^{iuM_i} - 1\right)\right)\right].$$

The previous expectation can be computed using a numerical integration over the distribution of the factor V. This can be achieved for example using a Gaussian quadrature. The computation of the loss distribution can then be accomplished thanks to the inversion formula and some Fast Fourier Transform algorithm. Let us remark that the former approach can be adapted without extra complication when losses given default M_i, $i = 1,\dots,n$ are stochastic but (jointly) independent together with default times. This method is described in Gregory and Laurent (2003) or Laurent and Gregory (2005). Gregory and Laurent (2004) investigate a richer correlation structure in which credit references are grouped in several sectors. They specify an inter-sector and an intra-sector dependence structure based on a factor approach and show that the computation of the loss distribution can be performed easily using the FFT approach.

Recursion approaches

An alternative approach, based on recursions is discussed in Andersen, Sidenius and Basu (2003) and Hull and White (2004)[6].

The first step is to split up the support of the loss distribution into constant width loss units. The width u of each loss unit is chosen such that each potential loss given default M_i can be approximated by a multiple of u. The support of the loss distribution is thus turned into a sequence $l = 0, u, \ldots, n_{\max} u$ where $n_{\max} > n$ and $n_{\max} u$ corresponds to the maximal potential loss $\sum_{1 \le i \le n} M_i$. Clearly, the simplest case is associated with constant losses given default, for instance $M_i = \frac{1-\delta}{n}$ with $\delta = 40\%$ and $n = 125$ which is a reasonable assumption for standard tranches. We can then choose $n_{\max} = n$.

The second step is performed thanks to the conditional independence of default events given the factor V. The algorithm starts from the conditional loss distribution associated with a portfolio set up with only one name, then it performs the computation of the conditional loss distribution when another name is added, and so on. Let us denote by $q_t^k(i)$, $i = 0, \ldots, n$ the conditional probability that the loss is equal to iu in the k^{th} portfolio where names $1, 2, \ldots, k$ $(k \le n)$ have been successively added. Let us start with a portfolio set up with only name number 1 with conditional default probability $p_t^{1|V}$, then

$$\begin{cases} q_t^1(0) = 1 - p_t^{1|V}, \\ q_t^1(1) = p_t^{1|V}, \\ q_t^1(i) = 0, \ i > 1. \end{cases}$$

Assume now that $q_t^k(.)$ has been computed after successive inclusion of names $2, \ldots, k$ in the pool. We then add firm $k+1$ in the portfolio with conditional default probability $p_t^{k+1|V}$. The loss distribution of the $(k+1)^{\text{th}}$ portfolio can be computed with the following recursive relation:

$$\begin{cases} q_t^{k+1}(0) = \left(1 - p_t^{k+1|V}\right) q_t^k(0), \\ q_t^{k+1}(i) = \left(1 - p_t^{k+1|V}\right) q_t^k(i) + p_t^{k+1|V} q_t^k(i-1), \ i = 1, \ldots, k+1, \\ q_t^{k+1}(i) = 0, \ i > k+1. \end{cases}$$

In the new portfolio, the loss can be iu either by being iu in the original portfolio if firm $k+1$ has not defaulted or by being $(i-1)u$ if firm $k+1$ has defaulted. The required loss distribution is the one obtained after all names have been added in the

[6] Let us remark that similar recursion methods have first been investigated by actuaries to compute the distribution of aggregate claims within individual life models. Several recursion algorithms originated from White and Greville (1959) have been developed such as the Z-method or the Newton method based on development of the loss generating function.

pool. It corresponds to $q_t^n(i)$, $i = 0,\ldots,n$. Let us remark that even though intermediate loss distributions obviously depend on the ordering of names added in the pool, the loss distribution associated to the entire portfolio is unique.

The last step consists of computing the unconditional loss distribution using a numerical integration over the distribution of the factor V. It is straightforward to extend the latter method to the case of stochastic and name dependent recovery rates. However, one of the key issues is to find a loss unit u which both allows getting enough accuracy on the loss distribution and driving low computational time. Hull and White (2004) present an extension of the former approach in which computation efforts are focused on pieces of the loss distribution associated with positive CDO tranche cash flows, allowing the algorithm to cope with non-constant width loss subdivisions. Other extensions based on approximation methods are discussed by Peretyatkin (2006).

Other approximation methods used by actuaries in the individual life model have also been adapted to the pricing of CDO tranches. For example, De Prisco *et al.* (2005) investigate the compound Poisson approximation, Jackson *et al.* (2007) propose to approximate the loss distribution by a Normal Power distribution.

Glasserman and Suchintabandid (2007) propose an approximation method based on power series expansions. These expansions express a CDO tranche price in a multifactor model as a series of prices computed within an independent default time model, which are easy to compute.

A new method based on Stein's approximation has been developed recently by Jiao (2007) and seems to be more efficient than standard approximation methods. In practical implementation, the conditional loss distribution (conditional to the factor) can be approximated either by a Gaussian or a Poisson random variable. Then CDO tranche premiums can be computed in each case using an additional corrector term known in closed form.

When considering CDO tranches on standardized indices, it is sometimes convenient to consider a homogeneous credit portfolio. In that case, the computation of the loss distribution reduces to a simple numerical integration.

I.3 Factor models in the case of homogeneous credit risk portfolios

In the case of a homogeneous credit risk portfolio, all names have the same nominal E and the same recovery rate δ. Consequently, the aggregate loss is proportional to the number of defaults N_t, i.e. $L_t = E(1-\delta)N_t$. Let us moreover assume that default times $\tau_1,\ldots,\tau_n$ are exchangeable, i.e. any permutation of default times leads to the same multivariate distribution function. Particularly, it means that all names have the same marginal distribution function, say F.

As a consequence of de Finetti's theorem[7], default indicators $D_1,\dots,D_n$ are Bernoulli mixtures[8] at any time horizon t. There exists a random mixture probability $\tilde{p}_t$ such that conditionally on $\tilde{p}_t$, $D_1,\dots,D_n$ are independent. More formally, if we denote by ν_t the distribution function of $\tilde{p}_t$, then for all $k = 0,\dots,n$,

$$P(N_t = k) = \binom{n}{k}\int_0^1 p^k (1-p)^{(n-k)} \nu_t(dp).$$

As a result, the aggregate loss distribution has a very simple form in the homogeneous case. Its computation only requires a numerical integration over ν_t which can be achieved using a Gaussian quadrature. Moreover, it can be seen that the factor assumption is not restrictive at all in the case of homogeneous portfolios. Homogeneity of credit risk portfolios can be viewed as a reasonable assumption for CDO tranches on large indices, although this is obviously an issue with equity tranches for which idiosyncratic risk is an important feature. A further step is to approximate the loss on large homogeneous portfolios with the mixture probability itself.

I.4 Large portfolio approximations

As CDO tranches are related to large credit portfolios, a standard assumption is to approximate the loss distribution with the one of an "infinitely granular portfolio"[9]. This fictive portfolio can be viewed as the limit of a sequence of aggregate losses on homogeneous portfolios, where the maximum loss has been normalized to unity: $L_t^n = \frac{1}{n}\sum_{i=1}^{n} D_i$, $n \geq 1$.

Let us recall that when default indicators $D_1,\dots,D_n,\dots$ form a sequence of exchangeable Bernoulli random variables and thanks to de Finetti's theorem, the normalized loss L_t^n converges almost surely to the mixture probability $\tilde{p}_t$ as the number of names tends to infinity. $\tilde{p}_t$ is also called the large (homogeneous) portfolio approximation. In a factor framework where default times are conditionally independent given a factor V, it can be shown that the mixture probability $\tilde{p}_t$ coincides with the conditional default probability $P\left(\tau_i \leq t \middle| V\right)$[10]. In the credit risk context this idea was firstly put in practice by Vasicek (2002). This approximation has also been studied by Gordy and Jones (2003), Greenberg *et al.* (2004a), Schloegl and O'Kane (2005) for the pricing of CDO tranches. Burtschell *et al.* (2008) compare the prices of CDO tranches based on the large portfolio approximation and on exact

[7] Aldous (1984) textbook gives a general account of de Finetti's theorem and some straightforward consequences.

[8] One needs that the default indicators are part of an infinite sequence of exchangeable default indicators.

[9] This terminology is taken from the Basel II agreement as it is the standard approach proposed by the Basel committee to determinate the regulatory capital related to bank credit risk management.

[10] The proof relies on a generalization of the strong law of large number. See Vasicek (2002) for more details.

computations. The large portfolio approximation can also be used to compare CDO tranche premiums on finite portfolios.

I.5 Comparing different factor models

Exchangeability of default times is a nice framework to study the impact of dependence on CDO tranche premiums. We have seen that the factor approach is legitimate in this context and we have exhibited a mixture probability $\tilde{p}_t$ such that, given $\tilde{p}_t$, default indicators $D_1,\ldots,D_n$ are conditionally independent. Thanks to the theory of stochastic orders, it is possible to compare CDO tranche premiums associated with portfolios with different mixture probabilities. Let us compare two portfolios with default indicators $D_1,\ldots,D_n$ and $D_1^*,\ldots,D_n^*$ and with (respectively) mixture probabilities $\tilde{p}_t$ and $\tilde{p}_t^*$. If $\tilde{p}_t$ is smaller than $\tilde{p}_t^*$ in the convex order[11], then the aggregate loss associated with $\tilde{p}_t$, $L_t=\sum_{i=1}^n M_i D_i$ is smaller than the aggregate loss associated with $\tilde{p}_t^*$, $L_t^*=\sum_{i=1}^n M_i D_i^*$ in the convex order[12]. See Cousin and Laurent (2007) for more details about this comparison method. Then, it can be proved (see Burtschell et al. (2008)) that when the mixture probabilities increase in the convex order, $[0,b]$ equity tranche premiums decrease and $[a,100\%]$ senior tranche premiums increase[13].

II) A review of factor approaches to the pricing of CDOs

In the previous section, we stressed the key role played by the distribution of conditional probabilities of default when pricing CDO tranches. Loosely speaking, specifying a multivariate default time distribution amounts to specifying a mixture distribution on default probabilities. We thereafter review a wide range of popular default risk models – factor copulas models, structural, multivariate Poisson, and Cox process based models – through a meticulous analysis of their mixture distributions.

II.1 Factor copula models

In copula models, the joint distribution of default times is coupled to its one-dimensional marginal distributions through a copula function C[14]:

$$P(\tau_1\le t_1,\ldots,\tau_n\le t_n)=C\left(F_1(t_1),\ldots,F_n(t_n)\right).$$

[11] Let X and Y be two scalar integrable positive random variables. We say that X precedes Y in convex order if $E[X]=E[Y]$ and $E\left[(X-K)^+\right]\le E\left[(Y-K)^+\right]$ for all $K\ge 0$.

[12] Losses given default $M_1,\ldots,M_n$ must be jointly independent from $D_1,\ldots,D_n$ and $D_1^*,\ldots,D_n^*$.

[13] As for the mezzanine tranche $[a,b]$ with $0<a<b<1$, it is not possible to infer such a comparison result. For example, it is well known that the present value of a mezzanine tranche may not be monotonic with respect to the compound correlation.

[14] For an introduction to copula functions with applications to finance, we refer to Cherubini *et al.* (2004) textbook.

In such a framework, the dependence structure and the marginal distribution functions can be handled separately. Usually, the marginal default probabilities $F_i(t_i)$ are inferred from the credit default swap premiums on the different names. Thus, they appear as market inputs. The dependence structure does not interfere with the pricing of single name credit default swaps and is only involved in the pricing of correlation products such as CDO tranches. In the credit risk field, this approach has been introduced by Li (2000) and further developed by Schönbucher and Schubert (2001).

Factor copula models are particular copula models for which the dependence structure of default times follows a factor framework. More specifically, the dependence structure is driven by some latent variables $V_1, \ldots, V_n$. Each variable V_i is expressed as a bivariate function of a common systemic risk factor V and an idiosyncratic risk factor $\bar{V}_i$:

$$V_i = f\left(V, \bar{V}_i\right), \; i = 1, \ldots, n,$$

where V and $\bar{V}_i$, $i = 1, \ldots, n$ are assumed to be independent. In most applications, the specified function f, the factors V and $\bar{V}_i$, $i = 1, \ldots, n$ are selected such that latent variables V_i, $i = 1, \ldots, n$ form an exchangeable sequence of random variables. Consequently, $\bar{V}_i$, $i = 1, \ldots, n$ must follow the same distribution function, say $\bar{H}$. Eventually, default times are defined by $\tau_i = F_i^{-1}\left(H\left(V_i\right)\right)$[15] where F_i are the distribution functions of default times and H the marginal distribution of latent variables V_i, $i = 1, \ldots, n$. For simplicity, we will thereafter restrict to the case where the marginal distributions of default times do not depend upon the name in the reference portfolio and denote the common distribution function by F.

In a general copula framework, computation of loss distributions requires n successive numerical integrations. The main interest of factor copula approach lies in its tractability as computational complexity is related to the factor dimension. Hence, factor copula models have been intensely used by market participants. In the following, we will review some popular factor copula approaches.

Additive factor copulas

The family of additive factor copulas is the most widely used as far as the pricing of CDO tranches is concerned. In this class of models, the function f is additive and latent variables $V_1, \ldots, V_n$ are related through a dependence parameter ρ taking values in $[0,1]$:

$$V_i = \rho V + \sqrt{1-\rho^2}\,\bar{V}_i, \; i = 1, \ldots, n.$$

[15] Let us remark that default times in a factor copula model can be viewed as first passage times in a multivariate static structural model where V_i, $i = 1, \ldots, n$ correspond to some correlated asset values and where $F(t)$ drives the dynamics of the default threshold. In fact, default times can be expressed as $\tau_i = \inf\left\{t \geq 0 \mid V_i \leq H^{-1}\left(F_i(t)\right)\right\}$, $i = 1, \ldots, n$.

From what was stated in previous sections, the conditional default probability or mixture probability $\tilde{p}_t$ can be expressed as:

$$\tilde{p}_t = \bar{H}\left(\frac{-\rho V + H^{-1}\left(F(t)\right)}{\sqrt{1-\rho^2}}\right).$$

In most applications, V and $\bar{V}_i$, $i = 1,\ldots,n$ belong to the same class of probability distributions which is chosen to be closed under convolution.

The most popular form of the model is the so-called factor Gaussian copula which relies on some independent standard Gaussian random variables V and $\bar{V}_i$, $i = 1,\ldots,n$ and leads to Gaussian latent variables $V_1,\ldots,V_n$. It has been introduced by Vasicek (2002) in the credit risk field and is known as the multivariate probit model in statistics[16]. Thanks to its tractability, the one factor Gaussian copula has become the financial industry benchmark despite of some well known drawbacks. For example, it is not possible to fit all market quotes of standard CDO tranches of the same maturity. This deficiency is related to the so-called correlation skew.

An alternative approach is the Student-t copula which embeds the Gaussian copula as a limit case. It has been considered for credit risk issues by a number of authors, including Andersen *et al.* (2003), Embrechts *et al.* (2003), Frey and McNeil (2003), Mashal *et al.* (2003), Greenberg *et al.* (2004b), Demarta and McNeil (2005), Schloegl and O'Kane (2005). Nevertheless, the Student-t copula features the same deficiency as the Gaussian copula.

For this reason, a number of additive factor copulas such as the double-t copula (Hull and White (2004)), the NIG copula (Guegan and Houdain (2005)), the double-NIG copula (Kalemanova *et al*. (2007)), the double Variance Gamma copula (Moosbrucker (2006)) and the α-stable copula (Prange and Scherer (2006)) have been investigated. Other heavy-tailed factor copula models are discussed in Wang *et al.* (2007). For a comparison of factor copula approaches in terms of pricing of CDO tranches, we refer to Burtschell *et al.* (2008). We plot in Figure 4 the mixture distributions associated with some of the previous additive factor copula approaches. Let us recall that mixture distributions correspond to the loss distribution of large homogeneous portfolios (see Section I.4).

[16] The multivariate probit model is a popular extension of the linear regression model in statistics. For a description of the model with applications to econometrics, we refer the reader to Gourieroux (2000).

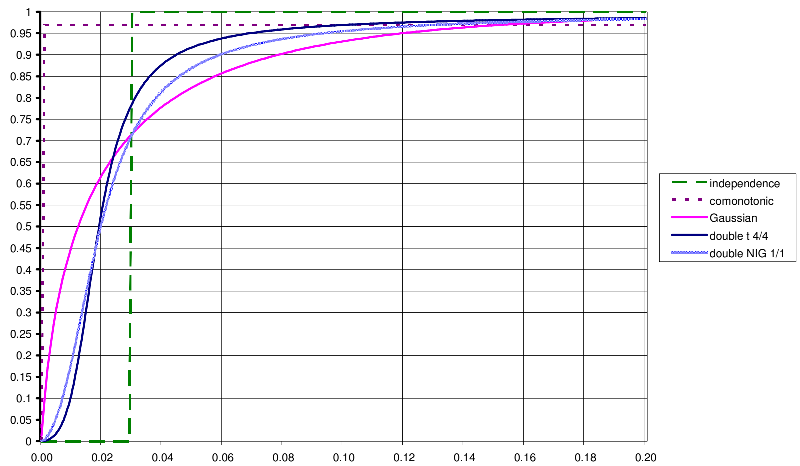

Figure 4 The graph shows the cumulative density functions of the mixture probability $\tilde{p}_t$ for the Gaussian, the double-t (4/4) and the double NIG (1/1) factor copula approaches. The marginal default probability is $F(t) = 2.96\%$ and we choose $\rho^2 = 30\%$ as the correlation between defaults. Eventually, we also plot the mixture distributions associated with the independence case ($\rho^2 = 0$) and the comonotonic case ($\rho^2 = 1$).

Stochastic correlation

Stochastic correlation models are other extensions of the factor Gaussian copula model. In this approach, the dependence parameter is stochastic. The latent variables are then expressed as:

$$V_i = \tilde{\rho}_i V + \sqrt{1-\tilde{\rho}_i^2}\,\overline{V}_i,\ i = 1,\ldots,n,$$

where V and $\overline{V}_i$, $i = 1,\ldots,n$ are independent standard Gaussian random variables and $\tilde{\rho}_i$, $i = 1,\ldots,n$ are identically distributed random variables taking values in $[0,1]$ and independent from V, $\overline{V}_i$, $i = 1,\ldots,n$. A suitable feature of this approach is that the latent variables V_i, $i = 1,\ldots,n$ follow a multivariate Gaussian distribution[17]. This eases calibration and implementation of the model.
Let us remark that in this framework, default times are exchangeable. Then, the conditional default probability $\tilde{p}_t$ can be expressed as:

$$\tilde{p}_t = \int_0^1 \Phi\left(\frac{-\rho V + \Phi^{-1}\left(F(t)\right)}{\sqrt{1-\rho^2}}\right) G\left(d\rho\right),$$

where G denotes the distribution function of $\tilde{\rho}_i$, $i = 1,\ldots,n$ and Φ is the Gaussian cumulative density function.

[17] Thanks to the independence between $\tilde{\rho}_i$, V, $\overline{V}_i$, $i = 1,\ldots,n$, given $\tilde{\rho}_i$, V_i follows a standard Gaussian distribution. Thus, after an integration over the distribution of $\tilde{\rho}_i$, the marginal distribution of V_i is also standard Gaussian.

Burtschell *et al.* (2008) investigated a two states stochastic correlation parameter. Tavares *et al*. (2004) also investigate a model with different states including a possibly catastrophic one. It has been shown by Burtschell *et al.* (2007) that a three states stochastic correlation model is enough to fit market quotes of CDO tranches for a given maturity. In their framework, the stochastic correlation parameters ρ_i, $i=1,\ldots,n$ have also a factor representation:

$$\tilde{\rho}_i = (1-B_s)(1-B_i)\rho + B_s$$

where B_s, $B_1,\ldots,B_n$ are independent Bernoulli random variables independent from V, $\bar{V}_i$, $i=1,\ldots,n$. Consequently, if we denote by $p_s = P(B_s=1)$ and $p = P(B_i=1)$, $i=1,\ldots,n$, default times are comonotonic ($V_i = V$) with probability p_s, independent ($V_i = \bar{V}_i$) with probability $(1-p_s)p$ and have a standard Gaussian factor representation with probability $(1-p_s)(1-p)$.

Mean-variance Gaussian mixtures

In this class of factor models, latent variables are simply expressed as mean-variance Gaussian mixtures:

$$V_i = m(V) + \sigma(V)\bar{V}_i,\ i=1,\ldots,n,$$

where V and $\bar{V}_i$, $i=1,\ldots,n$ are independent standard Gaussian random variables. Two popular CDO pricing models have been derived from this class, namely the random factor loading and the local correlation model.

The **random factor loading** model has been introduced by Andersen and Sidenius (2005b). In this approach, latent variables are modelled by:

$$V_i = m + \left(l1_{\{V<e\}} + h1_{\{V\geq e\}}\right)V + \nu\bar{V}_i,\ i=1,\ldots,n,$$

where l, h, e are some input parameters such that $l,h>0$. m and ν are chosen such that $E[V_i]=0$ and $E\left[V_i^2\right]=1$. This can be seen as a random factor loading model, since the risk exposure $l1_{\{V<e\}} + h1_{\{V\geq e\}}$ is state dependent. It is consistent with empirical researches showing that default correlation changes with respect to some macroeconomic random variables (see Das *et al.* (2006) and references therein). The conditional default probability can be written as:

$$\tilde{p}_t = \Phi\left(\frac{1}{\nu}\left(H^{-1}(F(t)) - m - \left(l1_{\{V<e\}} + h1_{\{V\geq e\}}\right)V\right)\right),$$

where H is the marginal distribution function of latent variables V_i, $i=1,\ldots,n$. Let us remark that contrary to the previous approaches, latent variables here are not Gaussian and the distribution function H depends upon the model parameters.

We compare in Figure 5 the mixture distribution functions obtained under a random factor loading model and a three states stochastic correlation model.

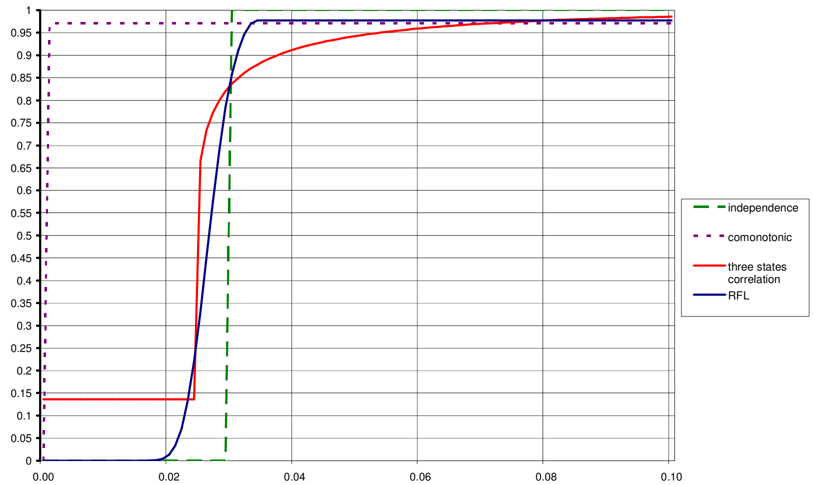

Figure 5 The graph shows the mixture distribution functions associated with the three states stochastic correlation model of Burtschell *et al.* (2007) and the random factor loading model of Andersen and Sidenius (2005b). The marginal default probability, $F(t) = 2.96\%$ holds to be the same for both approaches. As for the stochastic correlation model, the parameters are respectively $p_s = 0.14$, $p = 0.81$, $\rho^2 = 58\%$. As for the random factor loading model, we took $l = 85\%$, $h = 5\%$ and $e = -2$. The graph also shows the mixture distribution functions associated with the independence and the comonotonic case.

Like the three states version of the stochastic correlation model, this approach has the ability to fit perfectly market quotes of standardized CDO tranche spreads for a given maturity.

The **local correlation** model proposed by Turc *et al.* (2005) is associated with the following parametric modelling of latent variables:

$$V_i = -\rho(V)V + \sqrt{1-\rho^2(V)}\bar{V}_i,\ i = 1,\ldots,n,$$

where V and $\bar{V}_i$, $i = 1,\ldots,n$ are independent standard Gaussian random variables and $\rho(.)$ is some function of V taking values in $[0,1]$. $\rho(.)$ is known as the local correlation function. The conditional default probabilities can be written as:

$$\tilde{p}_t = \Phi\left(\frac{\rho(V)V + H^{-1}(F(t))}{\sqrt{1-\rho^2(V)}}\right),$$

where H is the marginal distribution function of latent variables V_i, $i = 1,\ldots,n$.

The local correlation can be used in a way which parallels the local volatility modelling in the equity derivatives market. This consists in a non parametric calibration of $\rho(.)$ on market CDO tranche premiums. The local correlation function has the advantage to be a model based implied correlation when compared to some standard market practice such as the compound and the base correlation. Moreover, there is a simple relationship between $\rho(.)$ and market compound correlations implied from CDO tranchlets[18] (marginal compound correlation) as explained in Turc *et al.* (2005) or Burtschell *et al.* (2007). But the trouble with this approach is that the existence and uniqueness of a local correlation function is not guarantied given an admissible loss distribution function possibly inferred from market quotes.

Archimedean copulas

Archimedean copulas have been widely used in credit risk modelling as they represent a direct alternative to the Gaussian copula approach. In most cases, there exists an effective and tractable way of generating random vectors with this dependence structure. Moreover, Archimedean copulas are inherently exchangeable and thus admit a factor representation. Marshall and Olkin (1988) first exhibit this factor representation in their famous simulation algorithm. More precisely, each Archimedean copula can be associated with a positive random factor V with inverse Laplace transform $\varphi(.)$ (and Laplace transform $\varphi^{-1}(.)$). In this framework, the latent variables can be expressed as:

$$V_i = \varphi^{-1}\left(\frac{-\ln \bar{V}_i}{V}\right),\ i = 1,\dots,n,$$

where $\bar{V}_i$, $i = 1,\dots,n$ are independent uniform random variables. Then, the joint distribution of the random vector $(V_1,\dots,V_n)$ is the φ-Archimedean copula[19]. In particular, each latent variable is a uniform random variable. Then the conditional default probability can be written as:

$$\tilde{p}_t = \exp\left(-\varphi\left(F(t)\right)V\right).$$

Let us remark that the previous framework corresponds to frailty models in the reliability theory or survival data analysis[20]. In these models, V is called a frailty since low levels of V are associated with shorter survival default times. The most popular Archimedean copula is probably the Clayton copula. In a credit risk context, it has been considered by, among others, Schönbucher and Schubert (2001), Gregory and Laurent (2003), Laurent and Gregory (2003), Madan *et al.* (2004), Friend and Rogge (2005). In addition, Rogge and Schönbucher (2003), Schloegl and O'Kane (2005) have investigated other Archimedean copulas such as the Gumbel or the Frank copula.

[18] CDO tranches $[a, a+1\%]$ with $0 \le a < 1$

[19] A random vector $(V_1,\dots,V_n)$ follows a φ-Archimedean copula if for all $v_1,\dots,v_n$ in $[0,1]^n$:

$$P(V_1 \le v_1,\dots,V_n \le v_n) = \varphi^{-1}\left(\varphi(v_1)+\dots+\varphi(v_n)\right).$$

[20] We refer the reader to Hougaard (2000) textbook for an introduction to multivariate survival data analysis and a detailed description of frailty models.

Copula	*Generator* φ	*Parameter*
Clayton	$t^{-\theta}-1$	$\theta \geq 0$
Gumbel	$\left(-\ln(t)\right)^{\theta}$	$\theta \geq 1$
Frank	$-\ln\left[\left(1-e^{-\theta t}\right)/\left(1-e^{-\theta}\right)\right]$	$\mathbb{R}^*$

Table 1 Some examples of Archimedean copulas with their generators.

In Figure 6, we compare the mixture distribution functions associated with a Clayton copula and a Gaussian factor copula. The dependence parameter θ of the Clayton copula has been chosen to get the same equity tranche premiums as with the one factor Gaussian copula model.

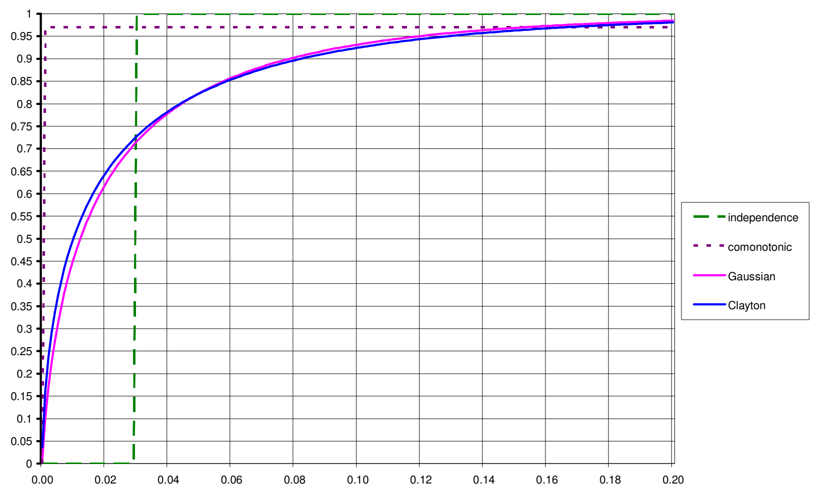

Figure 6 The graph shows the mixture distribution functions associated with a Clayton copula and a factor Gaussian copula. $F(t)=2.96\%$, $\rho^2=30\%$, $\theta=0.18$.

It can be seen that the distribution functions are very similar. Unsurprisingly, the resulting premiums for the mezzanine and senior tranches are also very similar in both approaches[21].

Perfect copula approach

As we saw in previous sections, much of the effort has focused on the research of a factor copula which best fits CDO tranche premiums. Let us recall that specifying a factor copula dependence structure is equivalent to specifying a mixture probability $\tilde{p}_t$. Hull and White (2006) exploit this remark and propose a direct estimation of the mixture probability distribution from market quotes. In their approach, for the sake of intuition on spread dynamics, the mixture probability is expressed through a hazard rate random variable $\tilde{\lambda}$ with a discrete distribution:

$$P\left(\tau_i \leq t \middle| \tilde{\lambda}=\lambda_k\right)=1-\exp\left(-\lambda_k t\right),\ k=1,\ldots,L.$$

[21] See Burtschell *et al.* (2008), Table 8, for more details about correspondence between parameters and assumptions on the underlying credit risk portfolio.

Then, defaults occur according to a mixture Poisson process (or a Cox Process) with hazard rate $\tilde{\lambda}$. Once a grid has been chosen for $\tilde{\lambda}$, the probability $\pi_k = P\left(\tilde{\lambda} = \lambda_k\right)$ can be calibrated in order to match market quotes of CDO tranches. Hull and White (2006) have shown that this last step is not possible in general. Consequently, they allow recovery rate to be a decreasing function of default rates, as suggested in some empirical researches such as Altman *et al.* (2005).

II.2 Multivariate structural models

Multivariate structural or firm value models are multi-name extensions of the so-called Black and Cox model where the firm default time corresponds to the first passage time of its asset dynamics below a certain threshold. This approach has first been proposed by Arvanitis and Gregory (2001) (chapter 5) in a general multivariate Gaussian setting for the pricing of basket credit derivatives. More recently, Hull *et al.* (2005) investigate the pricing of CDO tranche within a factor version of the Gaussian multivariate structural model. In the following, we follow the latter framework. We are concerned with n firms which may default in a time interval $\left[0,T\right]$. Their asset dynamics $V_1,\ldots,V_n$ are simply expressed as n correlated Brownian motions:

$$V_{i,t} = \rho V_t + \sqrt{1-\rho^2}\overline{V}_{i,t},\ i = 1,\ldots,n\ ,$$

where V , V_i , $i = 1,\ldots,n$ are independent standard Wiener processes. Default of firm i is triggered whenever the process V_i falls below a constant threshold a which is here assumed to be the same for all names. The corresponding default dates are then expressed as:

$$\tau_i = \inf\left\{t \geq 0 \middle| V_{i,t} \leq a\right\},\ i = 1,\ldots,n\ .$$

Clearly, default dates are independent conditionally on the process V . Let us remark that as the default indicators are exchangeable, the existence of a mixture probability is guarantied thanks to the de Finetti's theorem. We are thus in a one factor framework, though the factor depends on the time horizon contrary to the factor copula case. No mixture distribution can be expressed in closed form in the multivariate structural model. But, it is still possible to simulate losses on a large homogeneous portfolio (and then approximate the mixture probability $\tilde{p}_t$) in order to estimate the mixture distribution. Figure 7 shows that the latter happens to be very similar to the one generated within a factor Gaussian copula model. This is not surprising given the result of Hull *et al.* (2005) where CDO tranche premiums are very close in both frameworks. Moreover, the factor Gaussian copula can be seen as the static counterpart of the structural model developed above.

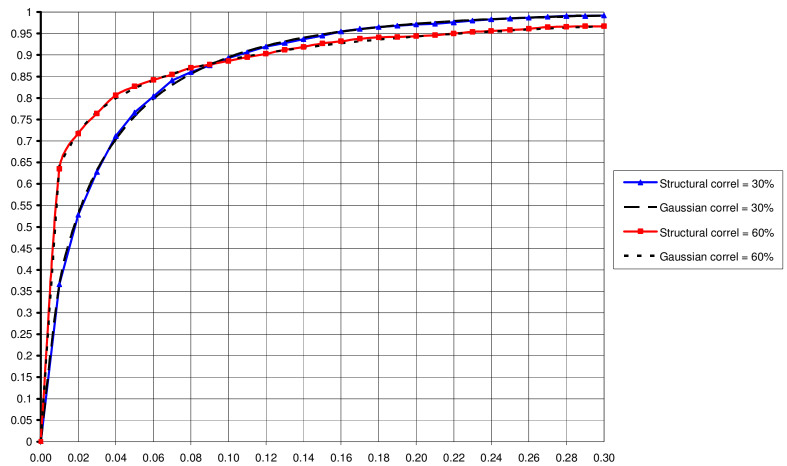

Figure 7 The graph shows empirical estimation of one year mixture distributions corresponding to structural models with correlation parameters $\rho^2 = 30\%$ and $\rho^2 = 60\%$. The barrier level is set at $a = -2$ such that the marginal default probability (before $t = 1$ year) is the same in both approaches and is equal to $F(t) = 3.94\%$. We then make a comparison with the mixture distribution associated with factor Gaussian copula models with the same correlation parameters and the same default probability.

The trouble with the first passage time models is that computation of CDO tranche premiums exclusively relies on Monte Carlo simulations and can be very time consuming. Kiesel and Scherer (2007) propose an efficient Monte Carlo estimation of CDO tranche spreads in a multivariate jump-diffusion setting. Other contributions such as Luciano and Schoutens (2006), Baxter (2007) and Willeman (2007) investigate the classical Merton model where default at a particular time t occurs if the value of assets is below the barrier at that particular point in time. In this framework, default indicators at time t are independent given the systemic asset value V_t and semi-analytical techniques as explained in part 1 can be used to compute CDO tranche premiums. Moreover, several empirical researches claim that the Merton structural model is a reasonable approximation of the more general Black-Cox structural model when considering the pricing of CDO tranches. Luciano and Schoutens (2006) consider a multivariate Variance Gamma model and show that it can be easily calibrated from market quotes. Baxter (2007) proposes to model the dynamics of assets with multivariate Lévy processes based on the Gamma process and Willeman (2007) investigate a multivariate structural model as in Hull *et al.* (2005) and adds a common jump component in the dynamic of assets.

II.3 Multivariate Poisson models

These models originate from the theory of reliability where they are also called shock models. In multivariate Poisson models, default times correspond to the first jump instants of a multivariate Poisson process $\left(N_t^1, \ldots, N_t^n\right)$. For example, when the Poisson process N_t^i jumps for the first time, it triggers the default of name i. The dependence between default events derives from the arrival of some independent systemic events or common shocks leading to the default of a group of names with a given probability. For the sake of simplicity, we limit ourselves to the case where only two independent shocks can affect the economy. In this framework, each default can be triggered either by an idiosyncratic fatal shock or by a systemic but not necessarily fatal shock. The Poisson process which drives default of name i can be expressed as:

$$N_t^i = \bar{N}_t^i + \sum_{j=1}^{N_t} B_j^i$$

where N_t and $\bar{N}_t^i$ are independent Poisson processes with respectively parameter λ and $\bar{\lambda}$[22]. We further assume that B_j^i, $i=1,\dots,n$, $j \geq 1$ are independent Bernoulli random variables with mean p independent of N_t and $\bar{N}_t^i$, $i=1,\dots,n$. Eventually, default times are described by:

$$\tau_i = \inf\left\{t \geq 0 \middle| N_t^i > 0\right\},\ i=1,\dots,n.$$

The background event (new jump of N_t) affects each name (independently) with probability p. A specificity of the multivariate Poisson framework is to allow for more than one default occurring in small time intervals. It also includes the possibility of some Armageddon phenomena where all names may default at the same time, then leading to fatten the tail of the aggregate loss distribution as required by market quotes. Let us stress that default dates are independent conditionally on the process N, while default indicators $D_1,\dots,D_n$ are independent given N_t.

By the independence of all sources of randomness, N_t^i, $i=1,\dots,n$ are Poisson processes with parameter $\bar{\lambda} + p\lambda$. As a result, default times are exponentially distributed with the same parameter. It can be shown that the dependence structure of default times is the one of the Marshall-Olkin copula (see Lindskog and McNeil (2003) or Elouerkhaoui (2006) for more details about this copula function). The Marshall-Olkin multivariate exponential distribution (Marshall and Olkin (1967)) has been introduced to the credit domain by Duffie and Singleton (1998) and also discussed by Li (2000) and Wong (2000). More recently, analytical results on the aggregate loss distribution have been derived by Lindskog and McNeil (2003) within a multivariate Poisson model. Some extensions are presented by Giesecke (2003), Elouerkhaoui (2006), Brigo *et al.* (2007a, 2007b).

In this multivariate Poisson model, default times and thus default indicators are exchangeable. The corresponding mixture probability can be expressed as:

$$\tilde{p}_t = 1 - (1-p)^{N_t} \exp\left(-\bar{\lambda} t\right).$$

As in the case of multivariate structural models, we are still in a one factor framework, where the factor depends on the time horizon. We plot in Figure 4, the distribution function associated to a Multivariate Poisson model. As the mixture probability is a discrete random variable, its distribution function is stepwise constant.

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[22] $\sum_{j=1}^{N_t} B_j^i$ is assumed to be equal to zero when $N_t = 0$.

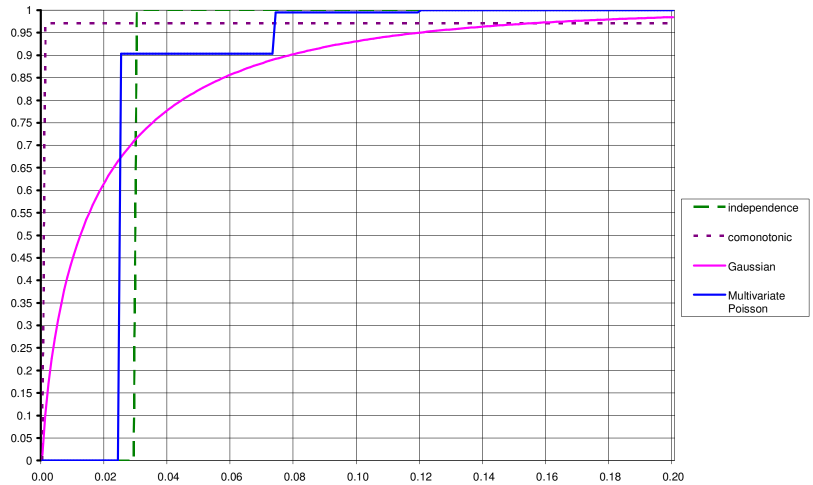

Figure 8 The graph shows the mixture distribution functions associated with a Multivariate Poisson model with $\bar{\lambda} = 0.5\%$, $\lambda = 2\%$ and $p = 5\%$. These parameters have been chosen such the marginal default probability before $t = 5$ years is $F(t) = 2.96\%$. For the sake of comparison, we also plot the mixture distribution function of the factor Gaussian copula with $\rho^2 = 30\%$.

II.4 Affine intensity models

In affine intensity models, the default date of a given name, say i, corresponds to the first jump time of a doubly stochastic Poisson process[23] with intensity λ_t^i. The latter follows an affine jump diffusion stochastic process which is assumed to be independent of the history of default times: there are no contagion effects of default events on the survival name intensities. Let us remark that, given the history of the process λ^i, survival distribution functions of default dates can be expressed as:

$$P\left(\tau_i \geq t \middle| \lambda_s^i,\ 0 \leq s \leq t\right) = \exp\left(-\int_0^t \lambda_s^i ds\right),\ i = 1, \ldots, n\,^{24}.$$

In affine models, dependence among default dates is concentrated upon dependence among default intensities. In the following, we follow the approach of Duffie and Gârleanu (2001) where the dependence among default intensities is driven by a factor representation:

$$\lambda_t^i = a x_t + x_t^i,\ i = 1, \ldots, n.$$

a is a non negative parameter accounting for the importance of the common factor and governing the dependence. The processes x, x^i, $i = 1, \ldots, n$ are assumed to be independent copies of an affine jump diffusion (AJD) process. The choice of AJD

[23] Also know as a Cox process.

[24] Conditionally on the history of default intensity λ_t^i, the default date τ_i is the first jump time of a non homogeneous Poisson process with intensity λ_t^i. Moreover, as far as simulations are concerned, default times are often expressed using some independent uniformly distributed random variables $U_1, \ldots, U_n$ independent of default intensities: $\tau_i = \inf\left\{ t \geq 0 \middle| \exp\left(-\int_0^t \lambda_s^i ds\right) \leq U_i \right\}$, $i = 1, \ldots, n$.

processes is not innocuous. First, the intensities λ_t^i, $i=1,\ldots,n$ remain in the class of AJD processes which allow to derive marginal default probabilities in closed form[25]. It results into a flexible dynamics of default intensities while letting the prospect for numerical implementations. Unlike copula models, this approach does not guarantee a perfect fit to CDS quotes for all maturities. Moreover, the same parameters drive the marginal distributions and the dependence structure of default times, which makes the calibration process more complicated.

Let us remark that default times are exchangeable in this framework. Moreover, conditionally on $V_t = \int_0^t x_s ds$, the default indicators $D_i = 1_{\{\tau_i \le t\}}$, $i=1,\ldots,n$ are independent. It is then possible to express the mixture probability $\tilde{p}_t$ associated with this exchangeable Bernoulli sequence:

$$\tilde{p}_t = P\left(\tau_i \le t \middle| x_s,\, 0 \le s \le t\right) = 1 - E\left[\exp\left(-\int_0^t x_s^i ds\right)\right]\exp\left(-a\int_0^t x_s ds\right).$$

As in the two previous examples, multivariate structural and Poisson models, we are in a one factor framework though a different factor is required to compute the loss distribution for each time horizon. Gregory and Laurent (2003) first exhibited the form of the mixture probability stressing the factor representation in affine models. Thanks to what stated above, it is possible to compute the characteristic function of $\tilde{p}_t$ and derive its density function using some inversion techniques. Mortensen (2006) and subsequently Eckner (2007) gradually extended the approach, providing more flexibility in the choice of parameters, and developed efficient numerical methods for the calibration and the pricing of CDO tranches. Chapovsky *et al*. (2007) provided a slightly different specification that guarantees a perfect calibration onto CDS quotes, but have to deal with positivity constraints on default intensities. Feldhütter (2007) performed an empirical analysis of the model using a large data set of CDS and CDO tranche spreads. He shows that when calibrated to daily CDS spreads, the model has a good ability to match marked-to-market of risky CDO tranche spreads over time while it does not capture properly the variability of senior tranches spreads.

Conclusion

The factor representation leads to efficient computational methods for the pricing of CDO tranches. It encompasses a wide range of CDO pricing models and also provides a suitable framework for portfolio risk analysis thanks to the theory of stochastic orders. Besides, when considering homogeneous credit risk portfolios, the factor approach is not restrictive thanks to de Finetti's theorem. We stressed the key role played by the mixture probability or the conditional default probability in factor models in terms of pricing CDO tranches and in deriving large portfolio approximations.

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[25] There exists some complex valued function $\alpha(.,.)$ and $\beta(.,.)$ depending on the process parameters such that $E\left[\exp\left(iu\int_0^t x_s ds\right)\right] = \exp\left(\alpha(u,t) + \beta(u,t)x_0\right)$. See Lando (2004) textbook for more details.

However, there are still a number of open questions to be dealt with among which we can mention:

- The calibration to CDO tranche quotes with different maturities and the same set of parameters is usually difficult.
- Whether one should choose a non parametric approach such as an implied copula or a properly specified parametric model is still unclear.
- Dealing with heterogeneity between names or linking factors related to different geographical regions or sectors, which is especially important for the pricing of bespoke CDOs.

Hopefully, there is still room for further improvements of the factor approach both on theoretical and practical grounds.

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