Hedging default risks of CDOs in Markovian contagion models

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Presentation related to paper:

Hedging default risks of CDOs in Markovian contagion models (2007)

Joint work with Jean-Paul Laurent and Jean-David Fermanian

Available on www.defaultrisk.com

Introduction

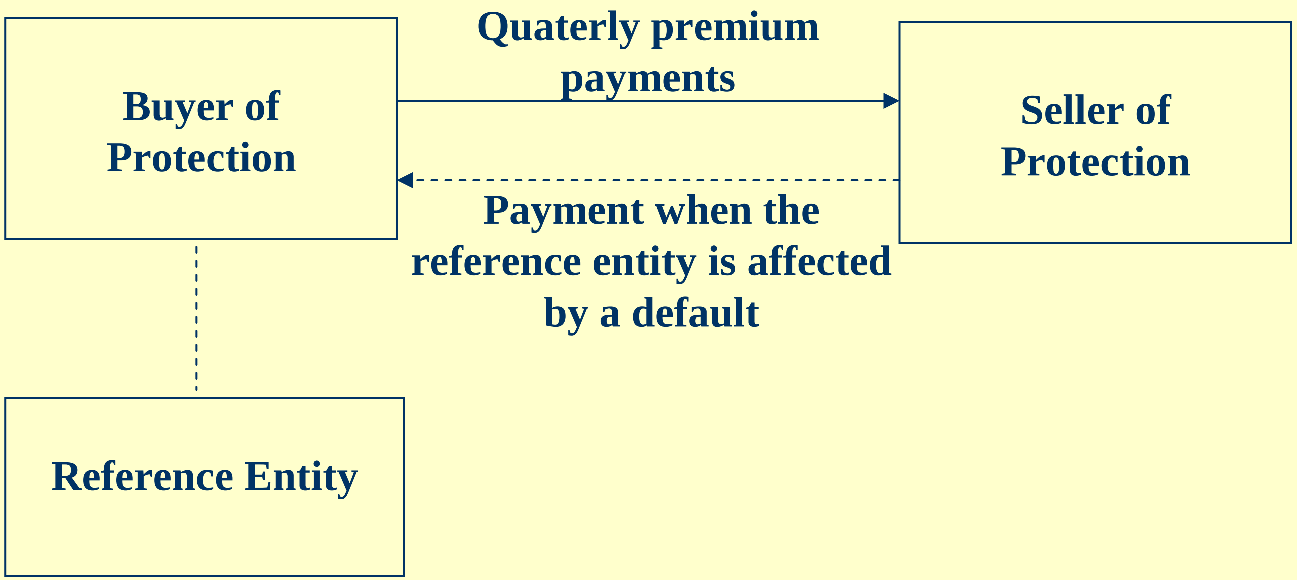
- In interest rate or equity markets, pricing is related to the cost of the hedge
  - ex: Black-Scholes pricing model, local volatility model
- In credit markets, pricing is disconnect from hedging
  - ex: The standard pricing model for CDO tranches does not rely on a replication argument
- Need to relate pricing and hedging

Introduction

- Purpose of the presentation
  - ➢ Focus on very specific aspects of default and credit spread risk
  - ➢ Under which the market for CDO tranches is complete
    - – **CDO tranches can be perfectly replicated by dynamically trading CDS (Credit Default Swaps)**
- Overlook of the presentation
  - ➢ Standardized CDO tranches
  - ➢ Tree approach to hedging defaults
    - – **Analogue of the local volatility model of Dupire (1994) or Derman & Kani (1994) for credit portfolio derivatives**
  - ➢ Results and comments
    - – **Hedging strategies obtained from a tree calibrated on market data**
    - – **Comparison with market practice**

Standardized CDO tranches

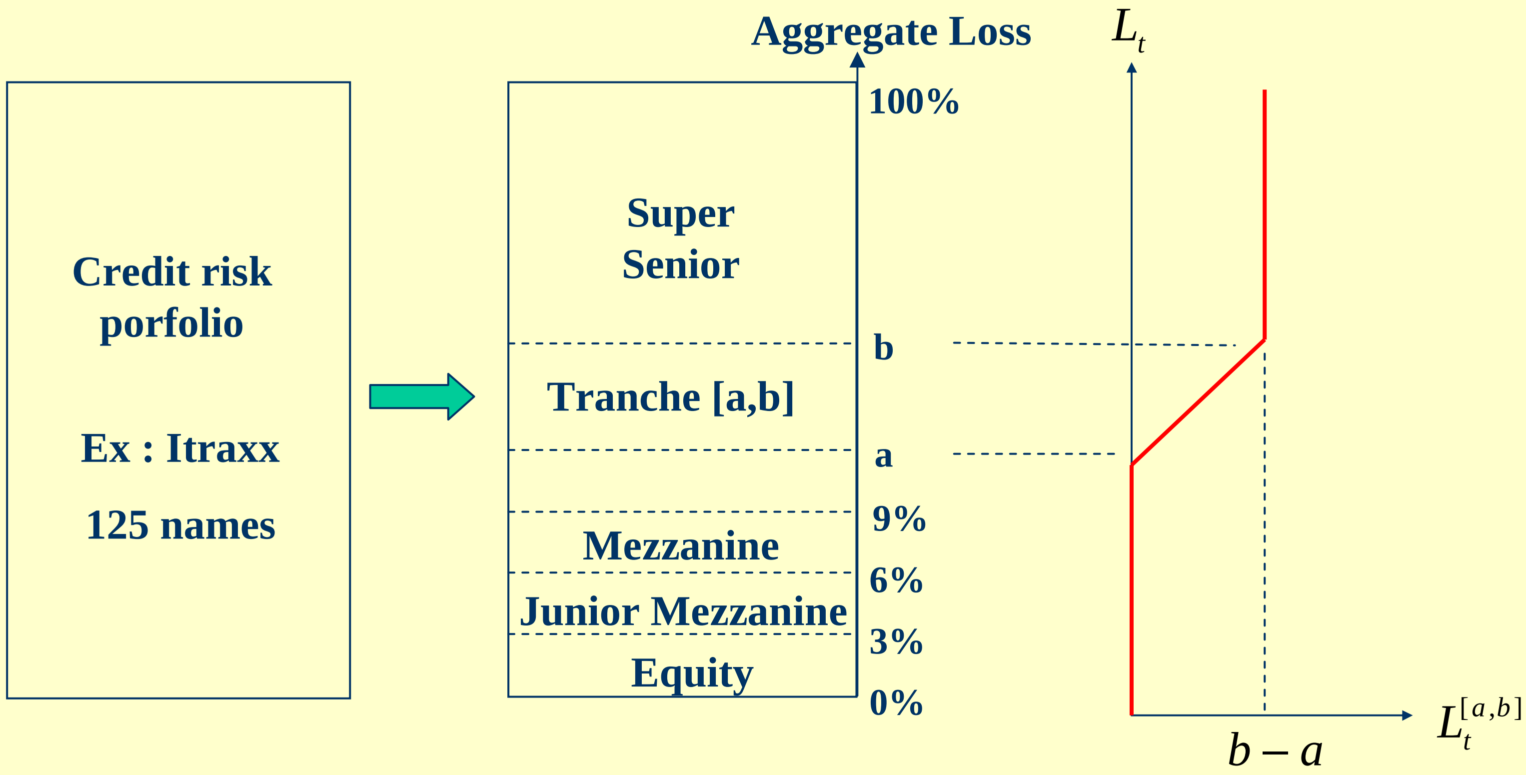
- What is a standardized CDO tranche ?
  - Bilateral contract between a buyer of protection and a seller of protection



  - The reference entity can be either
    - ➢ **Credit Default Swap Index (Itraxx Europe, CDX North America)**

Standardized CDO tranches

- What does a CDO tranche mean?



$$L_t^{[a,b]} = \left(L_t - a\right)^+ - \left(L_t - b\right)^+$$

Tree approach to hedging defaults

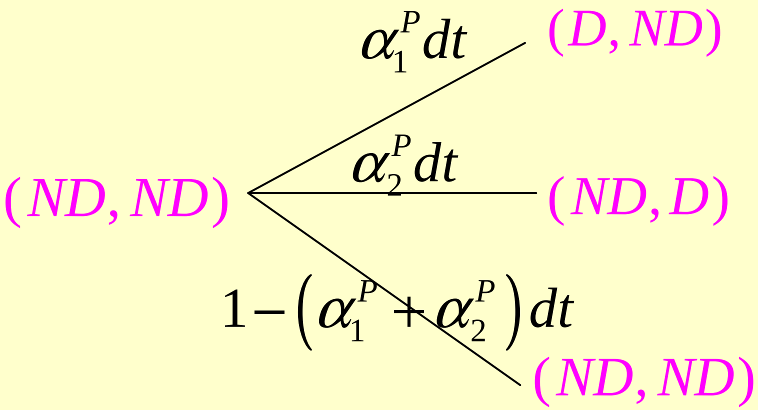
- We will start with two names only
  - Building a risk neutral tree of default states
  - Computation of prices along the tree for zero coupon CDO tranches
- Multiname case: homogeneous Markovian model
  - Building of risk-neutral tree for the aggregate loss
  - Computation of dynamic deltas
- Technical details can be found in the paper:
  - "hedging default risks of CDOs in Markovian contagion models"

Tree approach to hedging defaults

- Some notations :
  - τ_1, τ_2 default times of counterparties 1 and 2,
  - H_t available information at time t,
  - P historical probability,
  - α_1^P, α_2^P : (historical) default intensities:
    - $P\left[\tau_i \in [t, t+dt[\,\middle|\, H_t\right] = \alpha_i^P dt, \; i = 1,2$
- Assumption of « local » independence between default events
  - Probability of 1 and 2 defaulting altogether:
    - $P\left[\tau_1 \in [t, t+dt[, \tau_2 \in [t, t+dt[\,\middle|\, H_t\right] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } (dt)^2$
  - Local independence: simultaneous joint defaults can be neglected

Tree approach to hedging defaults

- Building up a tree:
  - Four possible states: (D,D), (D,ND), (ND,D), (ND,ND)
  - Under no simultaneous defaults assumption $p_{(D,D)}=0$
  - Only three possible states: (D,ND), (ND,D), (ND,ND)
  - Identifying (historical) tree probabilities:



$$
\begin{cases}
p_{(D,D)} = 0 \Rightarrow p_{(D,ND)} = p_{(D,D)} + p_{(D,ND)} = p_{(D,.)} = \alpha_1^P dt \\
p_{(D,D)} = 0 \Rightarrow p_{(ND,D)} = p_{(D,D)} + p_{(ND,D)} = p_{(.,D)} = \alpha_2^P dt \\
p_{(ND,ND)} = 1 - p_{(D,.)} - p_{(.,D)}
\end{cases}
$$

Tree approach to hedging defaults

- Stylized cash flows of **short term digital CDS on counterparty 1**:
  - CDS 1 premium $\alpha_1^Q dt$

0

- $\alpha_1^P dt$: $1-\alpha_1^Q dt$ (D, ND)
- $\alpha_2^P dt$: $-\alpha_1^Q dt$ (ND, D)
- $1-\left(\alpha_1^P+\alpha_2^P\right)dt$: $-\alpha_1^Q dt$ (ND, ND)

- Stylized cash flows of **short term digital CDS on counterparty 2**:
  - CDS 2 premium $\alpha_2^Q dt$

0

- $\alpha_1^P dt$: $-\alpha_2^Q dt$ (D, ND)
- $\alpha_2^P dt$: $1-\alpha_2^Q dt$ (ND, D)
- $1-\left(\alpha_1^P+\alpha_2^P\right)dt$: $-\alpha_2^Q dt$ (ND, ND)

Tree approach to hedging defaults

- Cash flows of **short term digital first to default swap** with premium $\alpha_F^Q dt$:

$$0 \begin{cases} \xrightarrow{\alpha_1^P dt} & 1-\alpha_F^Q dt & (D, ND) \\ \xrightarrow{\alpha_2^P dt} & 1-\alpha_F^Q dt & (ND, D) \\ \xrightarrow{1-(\alpha_1^P+\alpha_2^P)dt} & -\alpha_F^Q dt & (ND, ND) \end{cases}$$

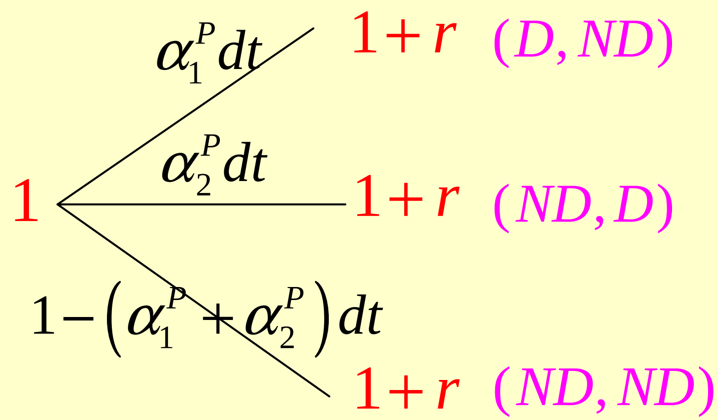
- Cash flows of holding CDS 1 + CDS 2:

$$0 \begin{cases} \xrightarrow{\alpha_1^P dt} & 1-(\alpha_1^Q+\alpha_2^Q)dt & (D, ND) \\ \xrightarrow{\alpha_2^P dt} & 1-(\alpha_1^Q+\alpha_2^Q)dt & (ND, D) \\ \xrightarrow{1-(\alpha_1^P+\alpha_2^P)dt} & -(\alpha_1^Q+\alpha_2^Q)dt & (ND, ND) \end{cases}$$

- Absence of arbitrage opportunities imply: $\alpha_F^Q = \alpha_1^Q + \alpha_2^Q$
- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2

Tree approach to hedging defaults

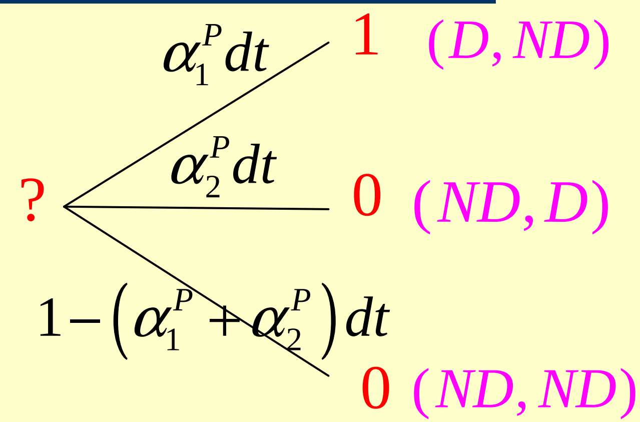
- Three possible states: (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
- Three tradable assets: CDS1, CDS2, risk-free asset

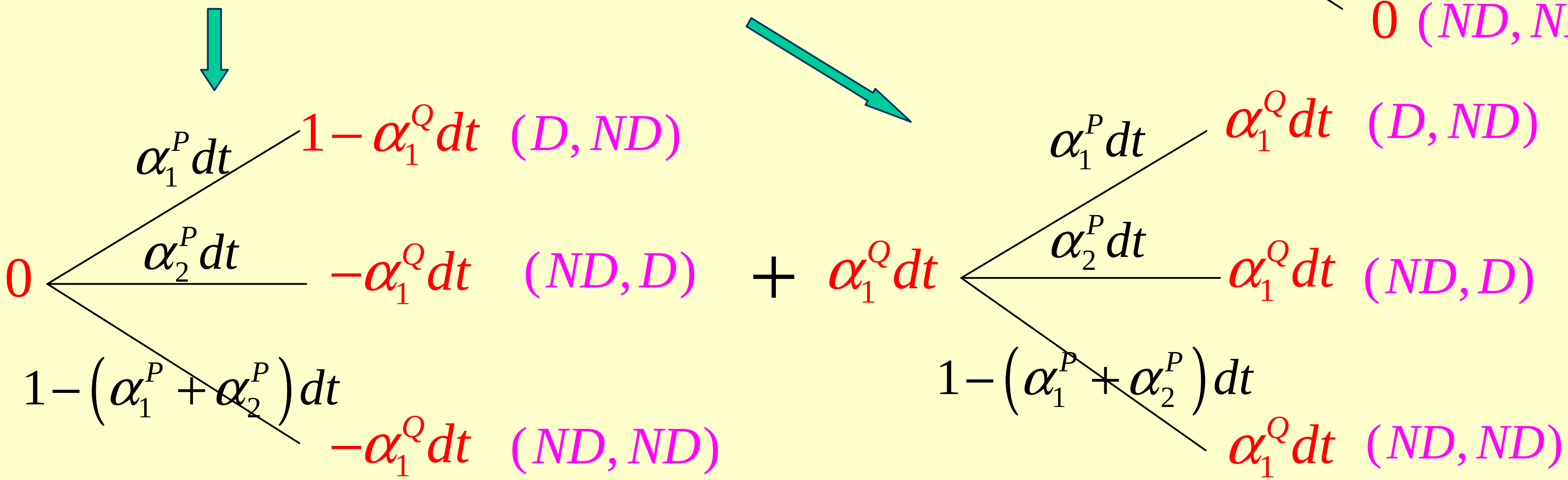


- For simplicity, let us assume $r=0$

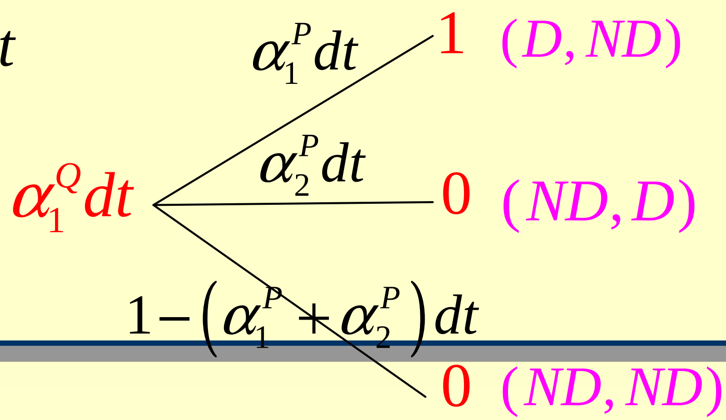
Tree approach to hedging defaults

- Three state contingent claims
  - Example: claim contingent on state (D, ND)
  - Can be replicated by holding
  - 1 CDS 1 + $\alpha_1^Q dt$ risk-free asset



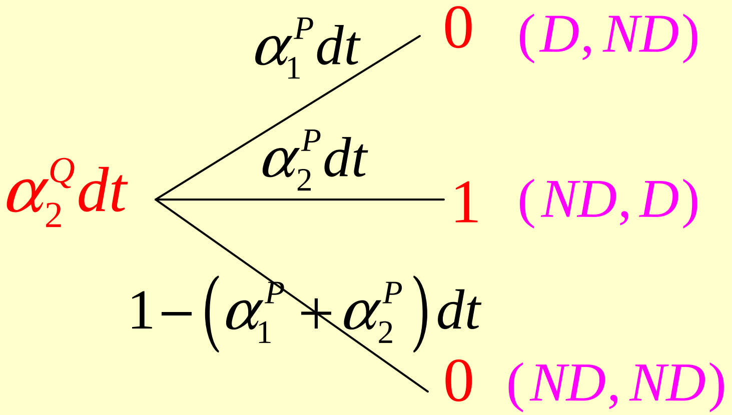


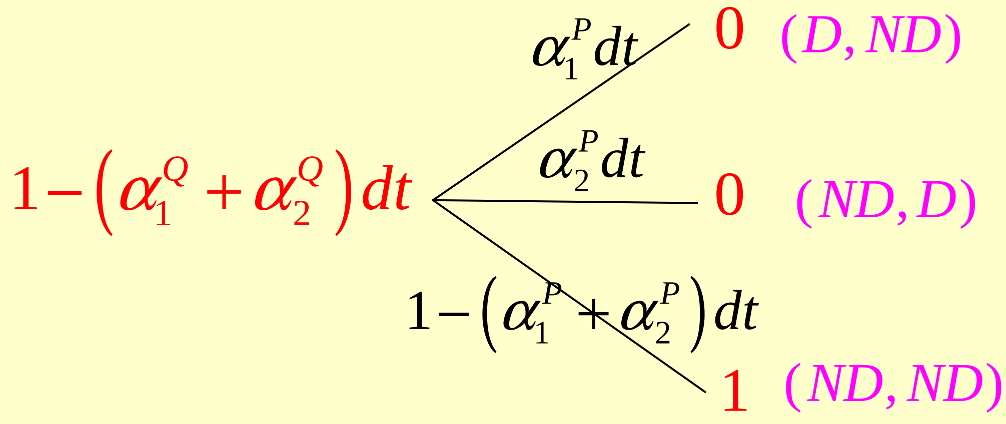
  - Replication price = $\alpha_1^Q dt$



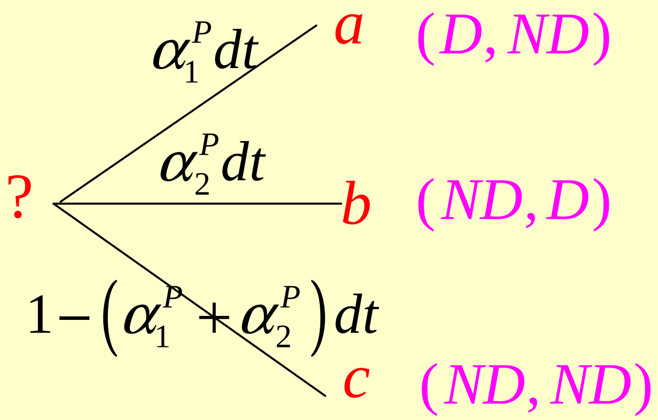
Tree approach to hedging defaults

- Similarly, the replication prices of the (ND, D) and (ND, ND) claims





- Replication price of:

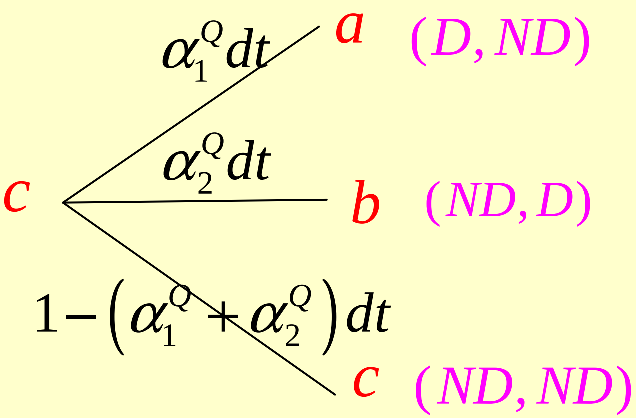


- Replication price = $\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1-(\alpha_1^Q+\alpha_2^Q)dt\right) \times c$

Tree approach to hedging defaults

- Replication price obtained by computing the expected payoff
  - Along a risk-neutral tree

$$\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1 - (\alpha_1^Q + \alpha_2^Q) dt\right) c$$



- Risk-neutral probabilities
  - Used for computing replication prices
  - Uniquely determined from short term CDS premiums
  - No need of historical default probabilities

Tree approach to hedging defaults

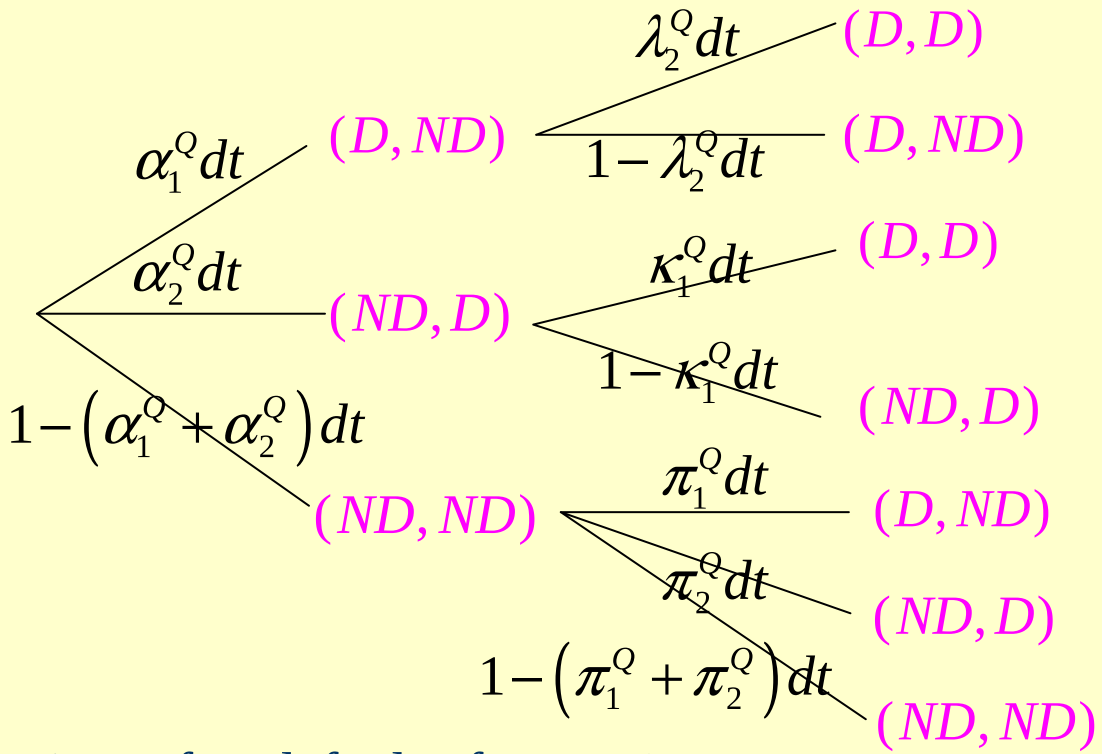
- Computation of deltas
  - Delta with respect to CDS 1: δ_1
  - Delta with respect to CDS 2: δ_2
  - Delta with respect to risk-free asset: p
    - ➢ p also equal to up-front premium

$$\begin{cases} a = p + \delta_1 \times \overbrace{\left(1 - \alpha_1^Q dt\right)}^{\text{payoff CDS 1}} + \delta_2 \times \overbrace{\left(-\alpha_2^Q dt\right)}^{\text{payoff CDS 2}} \\ b = p + \delta_1 \times \left(-\alpha_1^Q dt\right) + \delta_2 \times \left(1 - \alpha_2^Q dt\right) \\ c = p + \delta_1 \times \underbrace{\left(-\alpha_1^Q dt\right)}_{\text{payoff CDS 1}} + \delta_2 \times \underbrace{\left(-\alpha_2^Q dt\right)}_{\text{payoff CDS 2}} \end{cases}$$

  - As for the replication price, deltas only depend upon CDS premiums

Tree approach to hedging defaults

- Dynamic case:



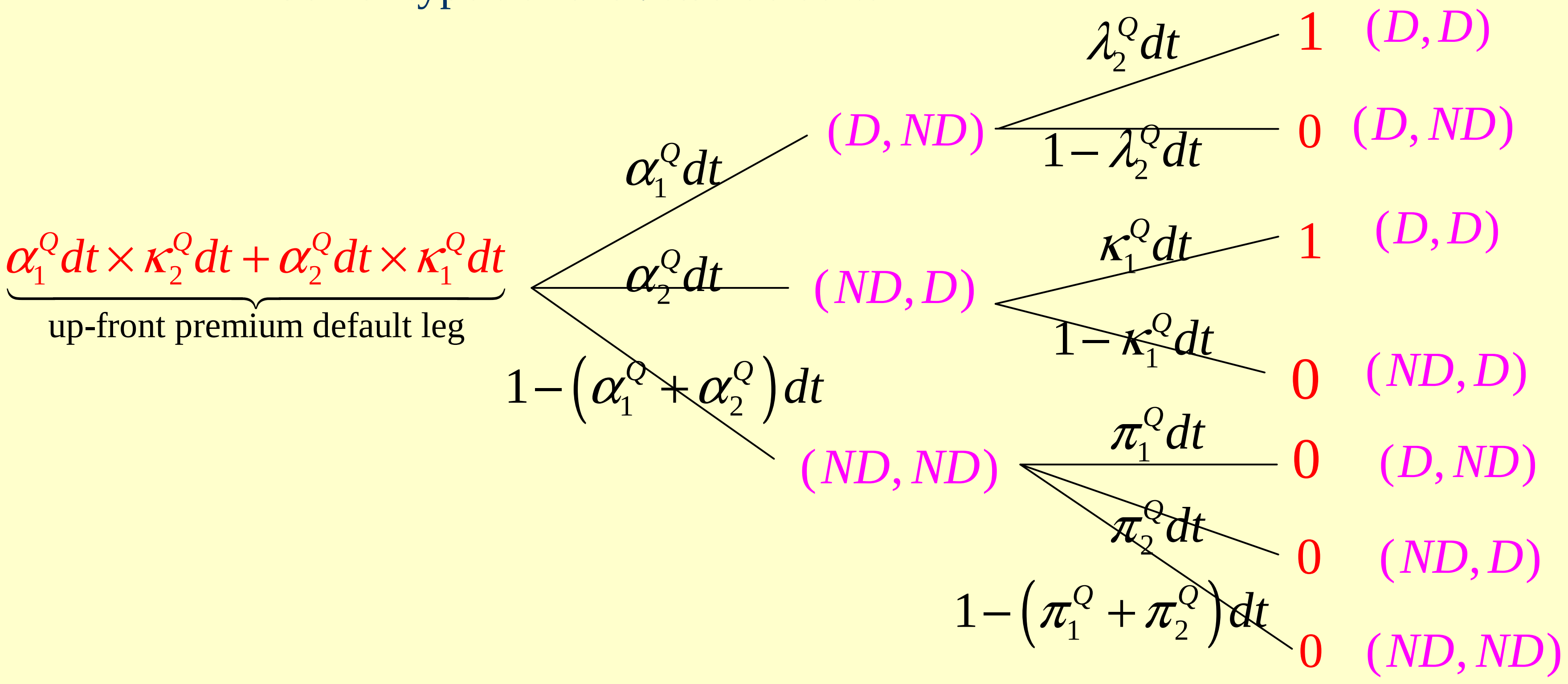
  - $\lambda_2^Q dt$ CDS 2 premium after default of name 1
  - $\kappa_1^Q dt$ CDS 1 premium after default of name 2
  - $\pi_1^Q dt$ CDS 1 premium if no name defaults at period 1
  - $\pi_2^Q dt$ CDS 2 premium if no name defaults at period 1
- Change in CDS premiums due to contagion effects
  - Usually, $\pi_1^Q < \alpha_1^Q < \kappa_1^Q$ and $\pi_2^Q < \alpha_2^Q < \lambda_2^Q$

Tree approach to hedging defaults

- Computation of prices and hedging strategies by backward induction
  - use of the dynamic risk-neutral tree
  - Start from period 2, compute price at period 1 for the three possible nodes
  - + hedge ratios in short term CDS 1,2 at period 1
  - Compute price and hedge ratio in short term CDS 1,2 at time 0

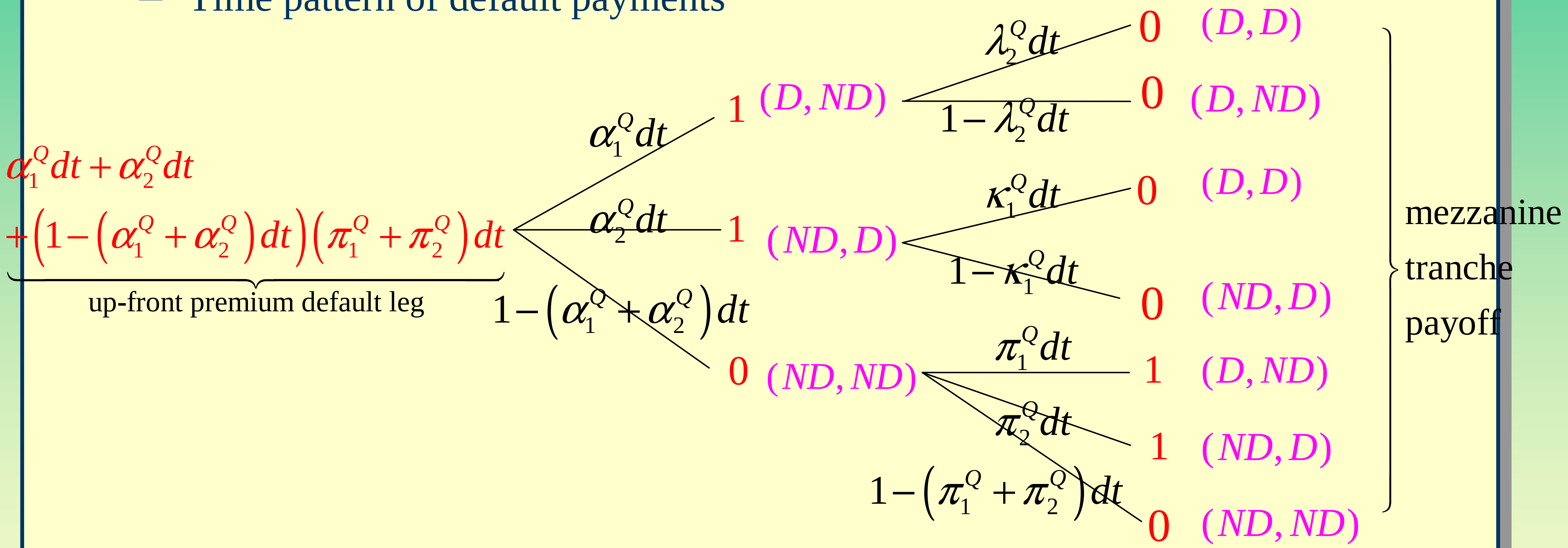
Tree approach to hedging defaults

- **Stylized example: default leg of a senior tranche**
  - Zero-recovery, maturity 2
  - Aggregate loss at time 2 can be equal to 0,1,2
    - Equity type tranche contingent on no defaults
    - Mezzanine type tranche : one default
    - Senior type tranche : two defaults



Tree approach to hedging defaults

- **Stylized example: default leg of a mezzanine tranche**
  - Time pattern of default payments



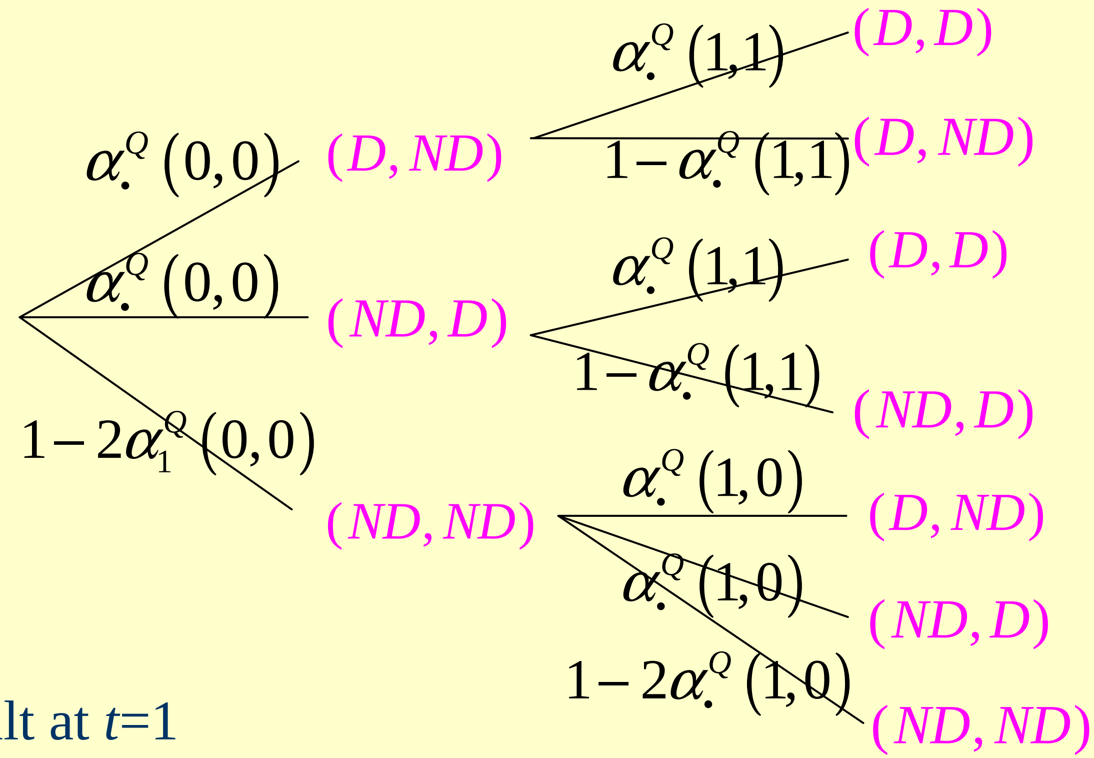
  - Possibility of taking into account discounting effects
  - The timing of premium payments
  - Computation of dynamic deltas with respect to short or actual CDS on names 1,2

Tree approach to hedging defaults

- In theory, one could also derive dynamic hedging strategies for standardized CDO tranches
  - Numerical issues: large dimensional, non recombining trees
  - Homogeneous Markovian assumption is very convenient
    - CDS premiums at a given time *t* only depend upon the current number of defaults $N(t)$
  - CDS premium at time 0 (no defaults) $\alpha_1^Q dt = \alpha_2^Q dt = \alpha_\bullet^Q \left(t = 0, N(0) = 0\right)$
  - CDS premium at time 1 (one default) $\lambda_2^Q dt = \kappa_1^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 1\right)$
  - CDS premium at time 1 (no defaults) $\pi_1^Q dt = \pi_2^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 0\right)$

Tree approach to hedging defaults

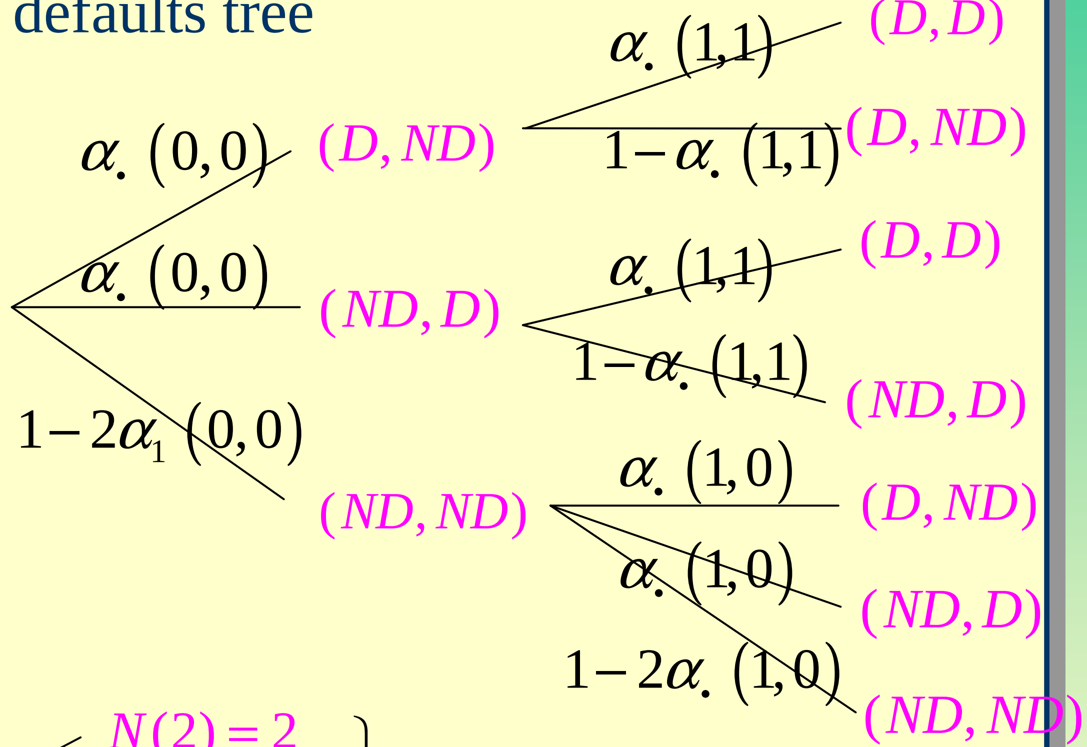
- Tree in the homogeneous case

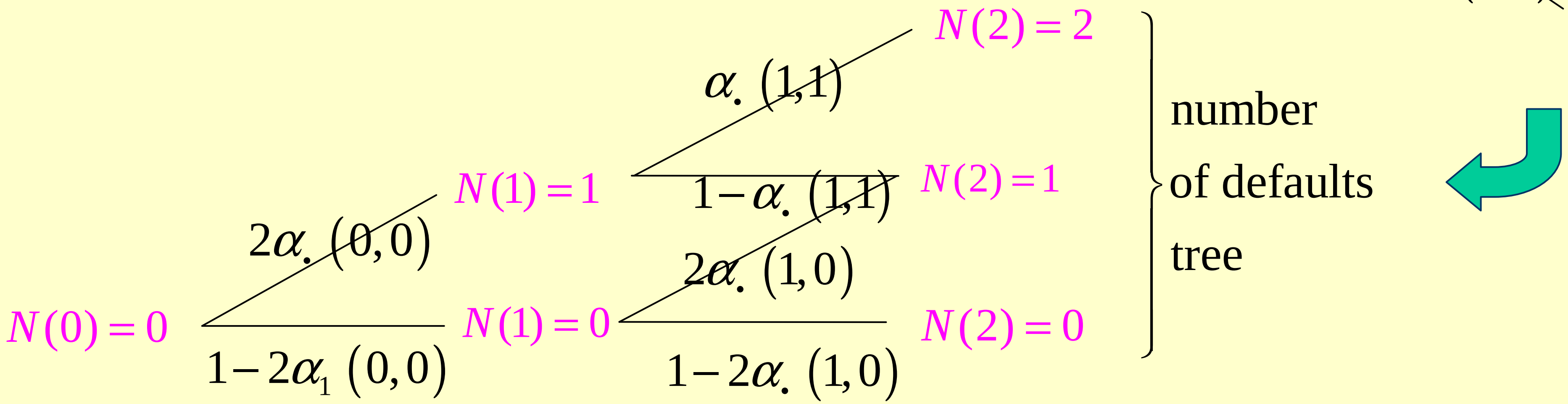


  - If we have $N(1)=1$, one default at t=1
  - The probability to have $N(2)=1$, one default at t=2…
  - Is $1-\alpha_{\cdot}^{Q}(1,1)$ and does not depend on the defaulted name at t=1
  - $N(t)$ is a **Markov process**
  - **Dynamics of the number of defaults can be expressed through a binomial tree**

Tree approach to hedging defaults

- From name per name to number of defaults tree

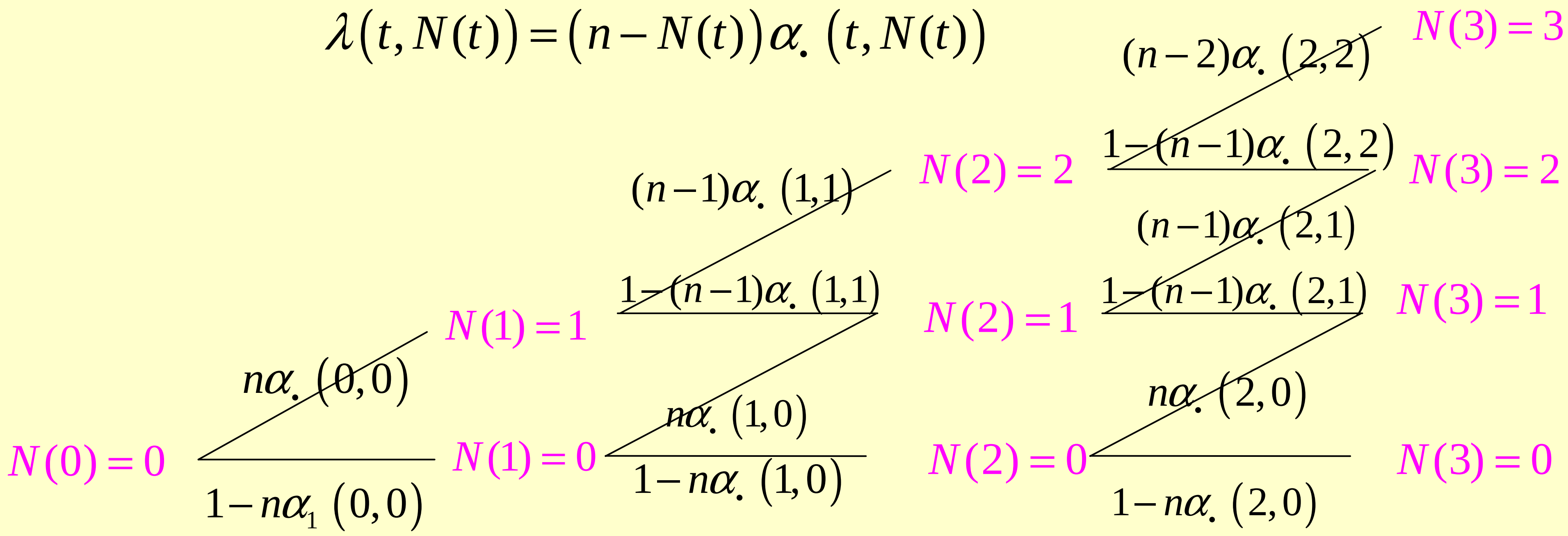




Tree approach to hedging defaults

- Easy extension to *n* names
  - Individual intensity at time *t* for $N(t)$ defaults: $\alpha_{\bullet}\left(t, N(t)\right)$
  - Number of defaults intensity : sum of surviving name intensities:

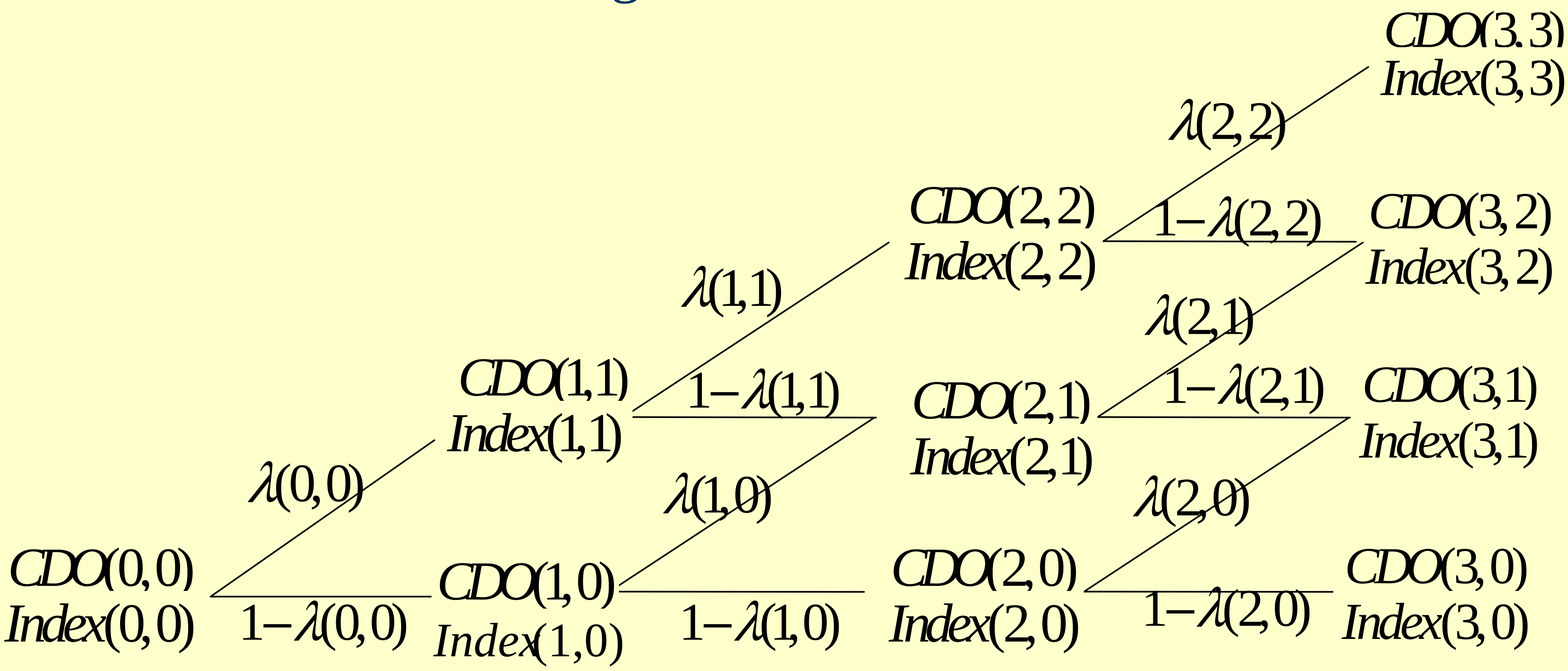
$$\lambda\left(t, N(t)\right) = \left(n - N(t)\right)\alpha_{\bullet}\left(t, N(t)\right)$$



  - $\alpha_{\bullet}\left(0,0\right), \alpha_{\bullet}\left(1,0\right), \alpha_{\bullet}\left(1,1\right), \alpha_{\bullet}\left(2,0\right), \alpha_{\bullet}\left(2,1\right), \ldots$ can be easily calibrated
  - on marginal distributions of $N(t)$ by forward induction.

Tree approach to hedging defaults

- Easy computation of CDO tranches and Index present values along the nodes of the tree using a backward induction



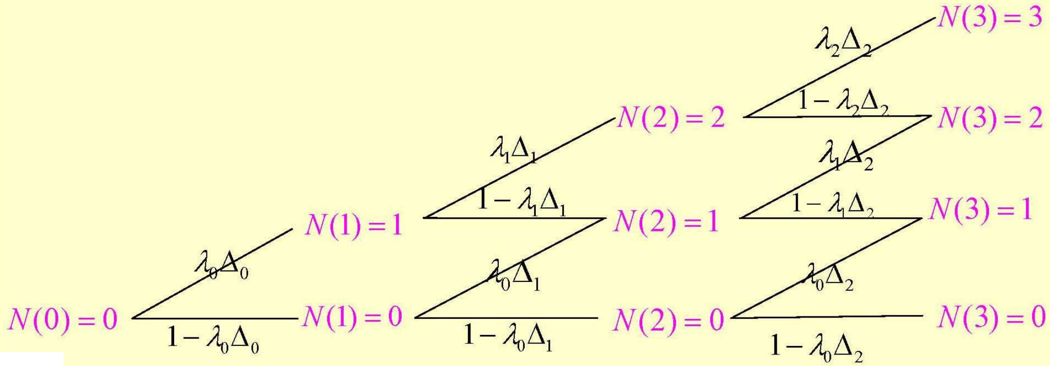
Tree approach to hedging defaults

- What about the credit deltas?
  - In a homogeneous framework, deltas with respect to CDS are all the same
  - Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  - Credit delta with respect to the credit default swap index
  - = change in PV of the tranche / change in PV of the CDS index

$$\delta\left(t, N(t)\right) = \frac{CDO\left(t+1, N(t)+1\right) - CDO\left(t+1, N(t)\right)}{Index\left(t+1, N(t)+1\right) - Index\left(t+1, N(t)\right)}$$

Results and comments

- Calibration of the tree on a market base correlation structure
  - Number of names: 125
  - Default-free interest rate: 4%
  - 5Y Credit spreads: 20bps
  - Recovery rate: 40%



3%	6%	9%	12%	22%
18%	28%	36%	42%	58%

Table 6. Base correlations with respect to attachment points.

- Loss intensities with respect to the number of defaults
  - For simplicity, assumption of time-homogeneous loss intensities
  - Increase in intensities: contagion effects
  - Compare flat and steep base correlation structures

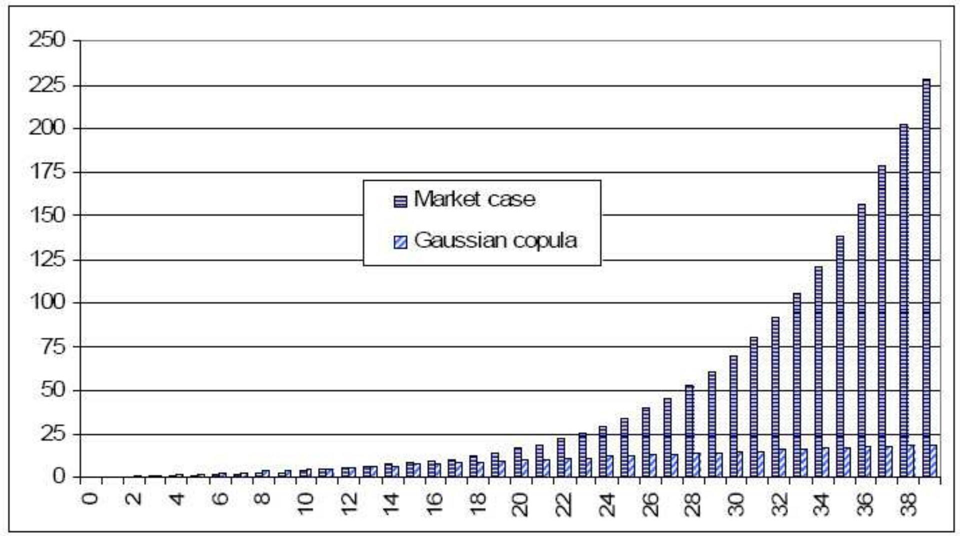


Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the x – axis.

Results and comments

- Dynamics of Credit Default Swap Index in the tree
  - In bps pa

Nb Defaults	Weeks			
	0	**14**	**56**	**84**
0	20	19	17	16
1	0	31	23	20
2	0	95	57	43
3	0	269	150	98
4	0	592	361	228
5	0	1022	723	490
6	0	1466	1193	905
7	0	1870	1680	1420
8	0	2243	2126	1945
9	0	2623	2534	2423
10	0	3035	2939	2859

  - The first default leads to a jump from 19bps to 31 bps
  - The second default is associated with a jump from 31 bps to 95 bps
  - Explosive behavior associated with upward base correlation curve

Results and comments

- Dynamics of credit deltas ([0,3%] equity tranche):

Nb Defaults	OutStanding Nominal	Weeks			
		0	**14**	**56**	**84**
0	**3.00%**	0.541	0.617	0.823	0.910
1	**2.52%**	0	0.279	0.510	0.690
2	**2.04%**	0	0.072	0.166	0.304
3	**1.56%**	0	0.016	0.034	0.072
4	**1.08%**	0	0.004	0.006	0.012
5	**0.60%**	0	0.002	0.002	0.002
6	**0.12%**	0	0.001	0.000	0.000
7	**0.00%**	0	0	0	0

  - Deltas are between 0 and 1
  - Gradually decrease with the number of defaults
    - ➢ **Equity tranche can be viewed as a short put position on the Index**
    - ➢ **Concave payoff, negative gammas**
  - When the number of defaults is > 6, the tranche is exhausted
  - Credit deltas increase in time
    - ➢ **Consistent with a decrease in time value**

Results and comments

- Comparison of market deltas and tree deltas (at inception)
  - Market delta computed under the standard Gaussian copula assumption
  - Base correlation is unchanged when shifting spreads ("correlation-sticky deltas")
  - Standard way of computing CDS index hedges in trading desks

	[0-3%]	[3-6%]	[6-9%]	[9-12%]	[12-22%]
Market deltas	27	4.5	1.25	0.6	0.25
Tree deltas	21.5	4.63	1.63	0.9	0.6

- Smaller equity tranche deltas in the tree model
  - How can we explain this?

Results and comments

- Smaller equity tranche deltas in the tree model (cont.)
  - Default is associated with an increase in dependence
    - ➢**Contagion effects**

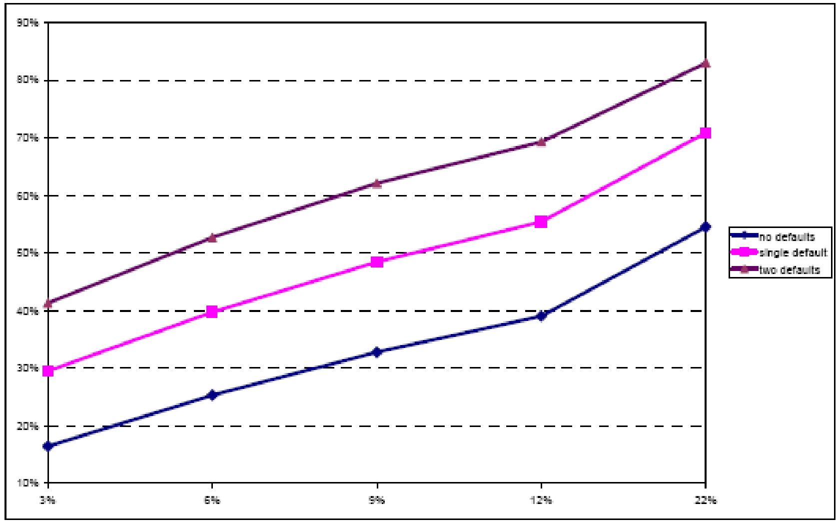


Figure 8. Dynamics of the base correlation curve with respect to the number of defaults. Detachment points on the x –axis. Base correlations on the y –axis.

- Increasing correlation leads to a decrease in the PV of the equity tranche
- Recent market shifts go in favor of the contagion model

Results and comments

- The current crisis is associated with joint upward shifts in credit spreads
  - ➢**Systemic risk**
- And an increase in base correlations

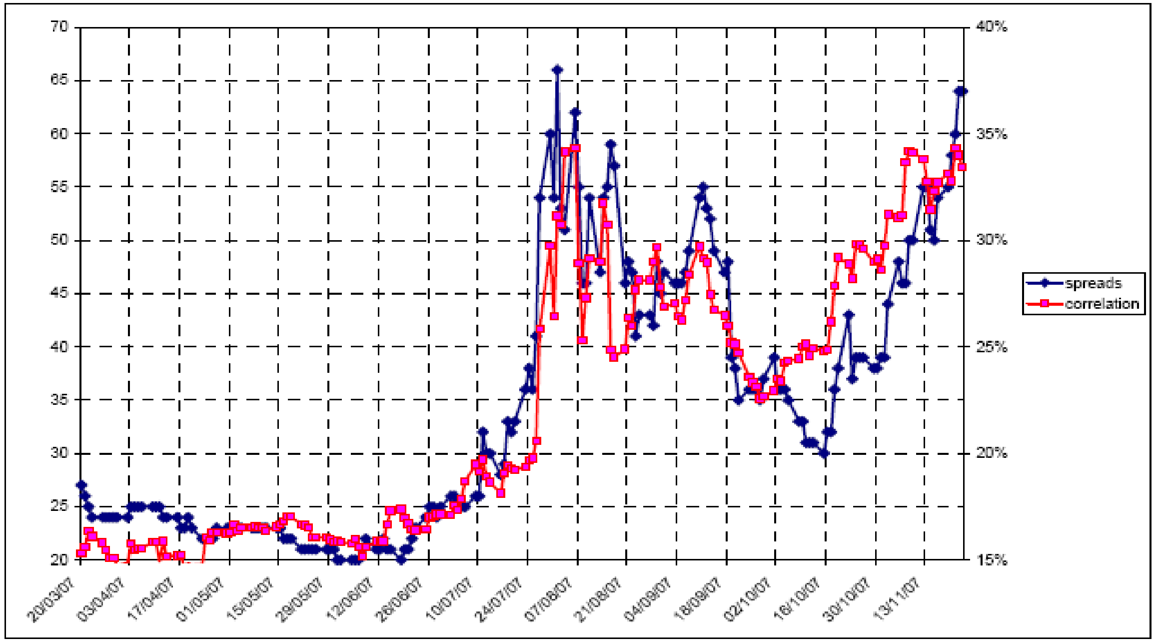


Figure 9. Credit spreads on the five years iTraxx index (Series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis

- Tree deltas are well suited in regimes of fear

Conclusion

- What do we learn from this hedging approach?
  - Thanks to stringent assumptions:
    - **credit spreads driven by defaults**
    - **homogeneity**
    - **Markov property**
  - It is possible to compute a dynamic hedging strategy
    - **Based on the CDS index**
  - That fully replicates the CDO tranche payoffs
    - **Model matches market quotes of liquid tranches**
    - **Very simple implementation**
    - **Credit deltas are easy to understand**
  - Improve the computation of default hedges
    - **Since it takes into account credit contagion**