Comparison results for credit risk portfolios

Areski COUSIN

Université Claude Bernard Lyon 1,
ISFA

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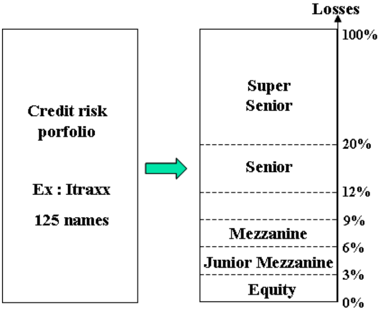
Joint work with Jean-Paul LAURENT

Introduction

- Presentation devoted to risk analysis of credit portfolios
- In credit risk portfolio modelling, dependence among default events is a crucial assumption
- We will investigate tranches of Collateralized Debt Obligation (CDO)
- Which is the impact of the dependence on
  - CDO tranche premiums ?
  - Risk measures on the aggregate loss ?

CDO tranches

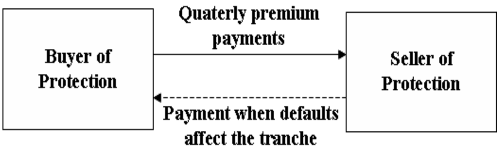
- Slice the credit portfolio into different risk levels or CDO tranches
- ex: CDO tranche on standardized Index such as CDX North America or Itraxx Europe



- $[0, 3\%]$ equity tranche is subordinated to $[3, 6\%]$ junior mezzanine tranche
- $[3, 6\%]$ junior mezzanine tranche is subordinated to $[6, 9\%]$ mezzanine tranche and so on,...

CDO tranches

- Each CDO tranche is a bilateral contract between a buyer of protection and a seller of protection:



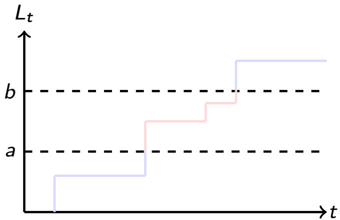
- CDO tranche cash flows are driven by the aggregate loss process

CDO tranches

- Credit portfolio with n reference entities
- $\tau_1, \dots, \tau_n$ default times
- $(D_1, \dots, D_n) = (1_{\{\tau_1 \le t\}}, \dots, 1_{\{\tau_n \le t\}})$ default indicators at time t
- $M_1, \dots, M_n$ losses given default assumed to be independent of default times
- Aggregate loss:

$$L_t = \sum_{i=1}^{n} M_i 1_{\{\tau_i \le t\}}$$

- Dynamics of the aggregate loss process:



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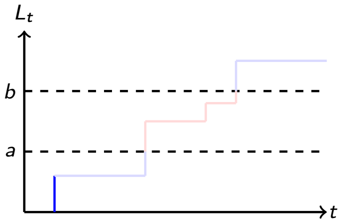

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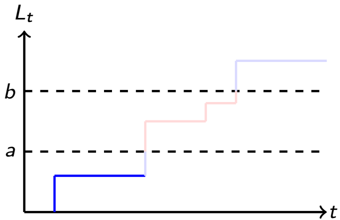

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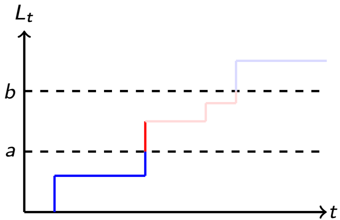

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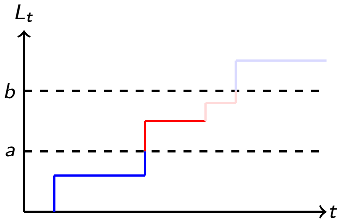

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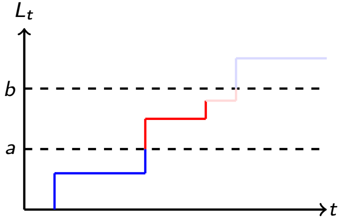

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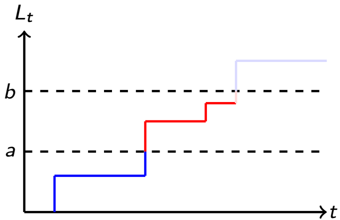

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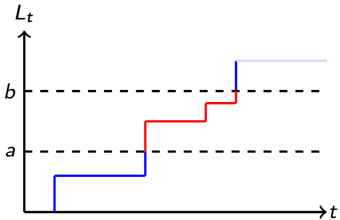

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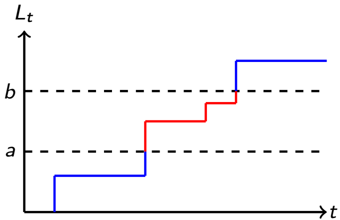
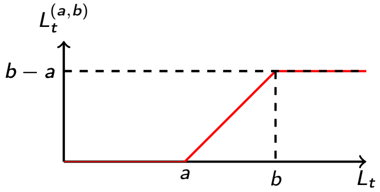


CDO tranches

- $L_t^{(a,b)}$ has a call spread payoff with respect to the aggregate loss:



- Loss on CDO tranche $[a, b]$:

$$L_t^{(a,b)} = (L_t - a)^+ - (L_t - b)^+$$

- Tranche premiums only involves call options on the aggregate loss L_t:

$$E\left[(L_t - a)^+\right] - E\left[(L_t - b)^+\right]$$

Motivation

- Specify the dependence structure of default indicators $D_1, \ldots, D_n$ which leads to:
  - an increase of the value of call options $E\left[(L_t - a)^+\right]$ for all strike level $a > 0$
  - an increase of convex risk measures on L_t (TVaR, Wang risk measures)
- Comparison between homogeneous credit portfolios
  - $D_1, \ldots, D_n$ are assumed to be exchangeable Bernoulli random variables
  - De Finetti Theorem leads to a factor representation
- Application to several default risk models

De Finetti theorem and factor representation

- Homogeneity assumption: default indicators $D_1, \ldots, D_n$ forms an exchangeable Bernoulli random vector

Definition (Exchangeability)

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \stackrel{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})r$$

De Finetti theorem and factor representation

- Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables
- Thanks to de Finetti theorem, there exists a random factor $\tilde{p}$ such that
- $D_1, \ldots, D_n$ are conditionally independent given $\tilde{p}$
- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum_i d_i}(1-p)^{n-\sum_i d_i} F_{\tilde{p}}(dp)$$

- $\tilde{p}$ is characterized by:

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \to \infty$$

Convex order

- The convex order compares the dispersion level of two random variables
- $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions f
- Particularly, if $X \leq_{cx} Y$ then $E[X] = E[Y]$ and $Var(X) \leq Var(Y)$
- Two important consequences of the convex order:
  - If $X \leq_{cx} Y$ then $E[(X - a)^+] \leq E[(Y - a)^+]$ for all $a > 0$
  - If $X \leq_{cx} Y$ then $\rho(X) \leq \rho(Y)$ for all law invariant and convex risk measures ρ (Bäuerle and Müller(2005))

Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- $(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)$ if $E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]$ for all supermodular function f

Definition (Supermodular function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in R^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$$

$$\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

- Consequences of new defaults are always worse when other defaults have already occurred
- If $(D_1, \ldots, D_n) \leq_{sm} (D_1, \ldots, D_n)$ then $\sum_{i=1}^n M_i D_i \leq_{cx} \sum_{i=1}^n M_i D_i$ (Müller(1997))

Main results

- Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L_t^* = \sum_{i=1}^{n} M_i D_i^*$
- Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixture factor $\tilde{p}$
- $D_1^*, \ldots, D_n^*$ exchangeable Bernoulli random variables associated with the mixture factor $\tilde{p}^*$

Theorem

$$\begin{aligned} \tilde{p} \leq_{cx} \tilde{p}^* \quad &\Rightarrow \quad (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \\ &\Rightarrow \quad E[(L_t - a)^+] \leq E[(L_t^* - a)^+] \textit{ for all } a > 0 \\ &\Rightarrow \quad \rho(L_t) \leq \rho(L_t^*) \textit{ for all convex risk measures } \rho \end{aligned}$$

Theorem

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \tag{1}$$

Additive factor copula approaches

- The dependence structure of default times is described by some latent variables $V_1, \ldots, V_n$ such that:
- $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i, \; i = 1 \ldots n$
- $V, \bar{V}_i, \; i = 1 \ldots n$ independent
- $\tau_i = G^{-1}(H_\rho(V_i)), \; i = 1 \ldots n$
    - G: distribution function of τ_i
    - H_ρ: distribution function of V_i
- $D_i = 1_{\{\tau_i \leq t\}}, \; i = 1 \ldots n$ are conditionally independent given V
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \leq t \mid V) = \tilde{p}$

Additive factor copula approaches

Theorem

For any fixed time horizon t, denote by $D_i = 1_{\{\tau_i \le t\}}$, $i = 1 \dots n$ and $D_i^* = 1_{\{\tau_i^* \le t\}}$, $i = 1 \dots n$ the default indicators corresponding to (resp.) ρ and ρ^*, then:

$$\rho \le \rho^* \Rightarrow \tilde{p} \le_{cx} \tilde{p}^* \Rightarrow (D_1, \dots, D_n) \le_{sm} (D_1^*, \dots, D_n^*)$$

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2005)
  - Double Variance Gamma: Moosbrucker(2005)

Structural model

Hull, Predescu and White(2005)

- Consider n firms
- Let $X_t^i, \ i = 1 \dots n$ be their asset dynamics

$$X_t^i = \rho W_t + \sqrt{1-\rho^2} W_t^i, \quad i = 1 \dots n$$

- $W, \ W^i, \ i = 1 \dots n$ are independent standard Wiener processes
- Default times as first passage times:

$$\tau_i = \inf\{t \in \boldsymbol{R}^+ | X_t^i \leq f(t)\}, \quad i = 1 \dots n, \ f : \boldsymbol{R} \to \boldsymbol{R} \text{ continuous}$$

- $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ are conditionally independent given $\sigma(W_t, \ t \in [0, T])$

Structural model

Theorem

For any fixed time horizon T*, denote by* $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \ldots n$ *and* $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \ldots n$ *the default indicators corresponding to (resp.)* ρ *and* ρ^**, then:*

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

Archimedean copula

Copula name	Generator φ	V-distribution
Clayton	$t^{-\theta} - 1$	$\text{Gamma}(1/\theta)$
Gumbel	$(-\ln(t))^{\theta}$	α-Stable, $\alpha = 1/\theta$
Frank	$-\ln\left[(1 - e^{-\theta t})/(1 - e^{-\theta})\right]$	Logarithmic series

Theorem

$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

- Other comparison results for multivariate Poisson models

Conclusion

- When considering homogeneous credit portfolios, the factor representation of default indicators is not restrictive
  - Thanks to De Finetti's theorem, there exists a mixture probability $\tilde{p}$ such that default indicators are conditionally independent given $\tilde{p}$
- This mixture probability is the key input to analyze the impact of dependence on:
  - CDO tranche premiums
  - Convex risk measures on the aggregate loss
- This analysis can be performed for several popular default risk models