Comparison results for homogenous credit portfolios

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Séminaire Transversal "Monnaie - Banque - Finance - Assurance"
ENS Cachan - 20 Décembre 2006

Contents

Exchangeability assumption

- n defaultable firms
- $\tau_1, \dots, \tau_n$ default times
- $(D_1, \dots, D_n) = (1_{\{\tau_1 \leq t\}}, \dots, 1_{\{\tau_n \leq t\}})$ default indicators
- Homogeneity assumption: default dates are assumed to be exchangeable

Definition (Exchangeability)

A random vector $(\tau_1, \dots, \tau_n)$ is exchangeable if its distribution function is invariant by permutation: $\forall \sigma \in S_n$

$$(\tau_1, \dots, \tau_n) \stackrel{d}{=} (\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)})$$

- Same marginals

De Finetti Theorem and Factor representation

- Suppose that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables
- There exists a random factor $\tilde{p}$ such that
- $D_1, \ldots, D_n$ are independent knowing $\tilde{p}$
- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum_i d_i}(1-p)^{n-\sum_i d_i} F_{\tilde{p}}(dp)$$

- $\tilde{p}$ is characterized by:

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \rightarrow \infty$$

Stochastic orders

- $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions f
- $X \leq_{sl} Y$ if $E[(X-K)^+] \leq E[(Y-K)^+]$ for all $K \in \boldsymbol{R}$
    - $X \leq_{sl} Y$ and $E[X] = E[Y] \Leftrightarrow X \leq_{cx} Y$
- $X \leq_{sm} Y$ if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions f

Definition (Supermodular function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **supermodular** if for all $x \in \boldsymbol{R}^n$, $1 \leq i < j \leq n$ and $\varepsilon, \delta > 0$ holds

$$f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n)$$

$$\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

- consequences of new defaults are always worse when other defaults have already occurred

Stochastic orders

- $(D_1, \dots, D_n)$ and $(D_1^* \dots, D_n^*)$ two exchangeable default indicator vectors
- M_i loss given default
- Aggregate losses:

$$L_t = \sum_{i=1}^{n} M_i D_i$$

$$L_t^* = \sum_{i=1}^{n} M_i D_i^*$$

Müller(1997)
Stop-loss order for portfolios of dependent risks.

$$(D_1, \dots, D_n) \leq_{sm} (D_1^* \dots, D_n^*) \Rightarrow L_t \leq_{sl} L_t^*$$

Stochastic orders

Theorem

Let $\mathbf{D} = (D_1, \ldots, D_n)$ *and* $\mathbf{D}^* = (D_1^*, \ldots, D_n^*)$ *be two exchangeable Bernoulli random vectors with (resp.)* F *and* F^* *as mixture distributions. Then:*

$$F \leq_{cx} F^* \quad \Rightarrow \quad \mathbf{D} \leq_{sm} \mathbf{D}^* \quad \text{and}$$

Theorem

Let $D_1, \ldots, D_n, \ldots$ *and* $D_1^*, \ldots, D_n^*, \ldots$ *be two exchangeable sequences of Bernoulli random variables. We denote by* F *(resp.* F^**) the distribution function associated with the mixing measure. Then,*

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow F \leq_{cx} F^*.$$

Multivariate Poisson model

Duffie(1998), Lindskog and McNeil(2003), Elouerkhaoui(2006)

- $\bar{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- N_t Poisson with parameter λ: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter p
- All sources of risk are independent
- $N_t^i = \bar{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \ i = 1 \dots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}, \ i = 1 \dots n$

Multivariate Poisson model

- $\tau_i \sim Exp(\bar{\lambda} + p\lambda)$
- $D_i = 1_{\{\tau_i \le t\}}, \; i = 1 \dots n$ are independent knowing N_t
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid N_t] = P(\tau_i \le t \mid N_t)$
- Conditional default probability:

$$\tilde{p} = 1 - (1-p)^{N_t} \exp(-\bar{\lambda} t)$$

Multivariate Poisson model

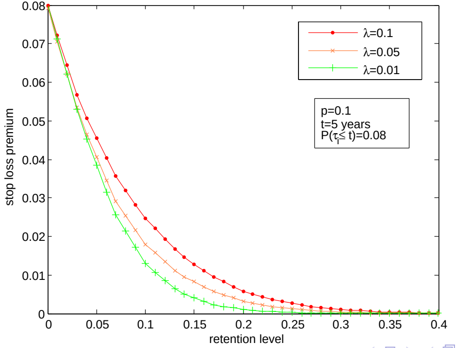
- Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$
- Supermodular order comparison requires equality of marginals: $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*$
- Comparison directions:
  - $p = p^*$: $\bar{\lambda}$ v.s λ
  - $\lambda = \lambda^*$: $\bar{\lambda}$ v.s p

Multivariate Poisson model

Theorem ($p = p^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ *and* $(\bar{\lambda}^*, \lambda^*, p^*)$ *be such that* $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$*, then:*

$$\lambda \leq \lambda^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$

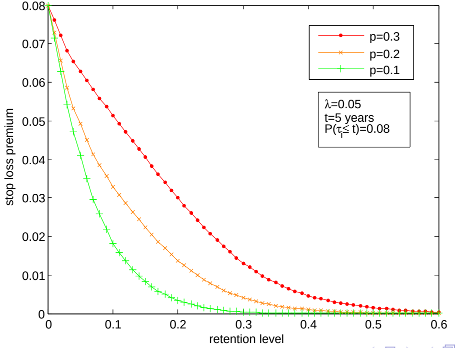


Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ *and* $(\bar{\lambda}^*, \lambda^*, p^*)$ *be such that* $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, *then:*

$$p \leq p^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



Structural Model

Hull, Predescu and White(2005)

- Consider n firms
- Let $X_t^i,\ i = 1 \ldots n$ be their asset dynamics

$$X_t^i = \rho W_t + \sqrt{1-\rho^2} W_t^i, \quad i = 1 \ldots n$$

- $W,\ W^i,\ i = 1 \ldots n$ are independent standard Wiener processes
- Default times as first passage times:

$$\tau_i = \inf\{t \in \boldsymbol{R}^+ | X_t^i \leq f(t)\}, \quad i = 1 \ldots n,\ f : \boldsymbol{R} \to \boldsymbol{R} \text{ continuous}$$

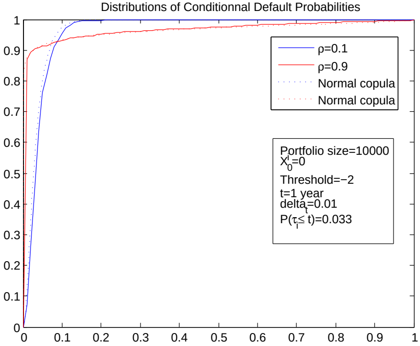
- $D_i = 1_{\{\tau_i \leq T\}}\ , i = 1 \ldots n$ are independent knowing $\sigma(W_t,\ t \in [0, T])$
- $\frac{1}{n} \sum_{i=1}^n D_i \xrightarrow{a.s} \tilde{p}$

Structural Model

Theorem

For any fixed time horizon T, denote by $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ and $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \dots n$ the default indicators corresponding to (resp.) ρ and ρ^*, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$



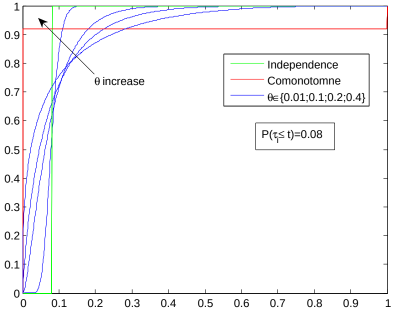
- $\tilde{p}(\rho) \leq_{cx} \tilde{p}(\rho^*)$

Archimedean copula

Copula name	Generator φ	V-distribution
Clayton	$t^{-\theta}-1$	Gamma$(1/\theta)$
Gumbel	$(-\ln(t))^{\theta}$	α-Stable, $\alpha=1/\theta$
Franck	$-\ln\left[(1-e^{-\theta t})/(1-e^{-\theta})\right]$	Logarithmic series

Theorem

$$\alpha \leq \alpha^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1,\ldots,D_n) \leq_{sm} (D_1^*,\ldots,D_n^*)$$



- $\tilde{p}(\theta) \leq_{cx} \tilde{p}(\theta^*)$

Additive copula framework

- $V_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i$
- $V, V_i \; i = 1 \ldots n$ independent
- Laws of $V, V_i \; i = 1 \ldots n$ do not depend on the dependence parameter ρ
- Standard copula models:
  - Gaussian, Student t
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2005)
  - Double Variance Gamma: Moosbrucker(2005)

Theorem

$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

Conclusion

- Characterization of supermodular order for exchangeable Bernoulli random vectors
- Comparison of CDO tranche premiums in several pricing models
- Unified way of presenting default risk models