Comparison results for homogenous credit portfolios

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Séminaire Lyon-Lausanne - 11 Décembre 2006

General background context

- n defaultable firms or policies
- $\tau_1, \ldots, \tau_n$ default times or claim occurrences
- $(D_1, \ldots, D_n) = (1_{\{\tau_1 \le t\}}, \ldots, 1_{\{\tau_n \le t\}})$ default or claim indicators
- M_i loss given default or claim amount
- Aggregate loss or total claim amount:

$$L_t = \sum_{i=1}^{n} M_i 1_{\{\tau_i \le t\}}$$

- Stop Loss order results for L_t?
- Ordering of convex risk measures on L_t?

Interest

- Specify the dependence structure between $D_1, \ldots, D_n$ which leads to:
  - an increase of stop loss premiums
  - an increase of convex risk measures
- Exchangeability of $D_1, \ldots, D_n$
  - De Finetti Theorem leads to a factor representation
  - Simplifies comparison analysis
- Comparison of Exchangeable Bernoulli random vectors
- Application to several models of default and insurance
  - Measure the impact of parameters governing the dependence
  - Comparing copula, structural, multivariate Poisson models

Contents

Exchangeability of default times

- Homogeneity assumption: default dates are assumed to be exchangeable

Definition (Exchangeability)

A random vector $(\tau_1, \ldots, \tau_n)$ is exchangeable if its distribution function is invariant by permutation: $\forall \sigma \in S_n$

$$(\tau_1, \ldots, \tau_n) \stackrel{d}{=} (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)})$$

- Same marginals

De Finetti Theorem and Factor representation

Theorem (De Finetti)

Suppose that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables, then there is a mixture probability measure ν such that $\forall n, \ \forall (d_1, \ldots, d_n) \in \{0,1\}^n$:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^{\sum_i d_i}(1-p)^{n-\sum_i d_i} d\nu(p)$$

- Usual De Finetti involves infinite sequences
  - Finite exchangeability only leads to a sign measure Jaynes 1986

De Finetti Theorem and Factor representation

- Denote by F the distribution function of ν: $F(p) = \nu\left(]0, p]\right)$
- There exists a random factor $\tilde{p}$ distributed as F such that:
- $D_1, \ldots, D_n$ are independent knowing $\tilde{p}$
- $\tilde{p}$ is a.s unique and such that:

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p} \quad \text{as} \quad n \longrightarrow \infty$$

Stochastic orders

- $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions f
- $X \leq_{icx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all increasing convex functions f
- $X \leq_{sl} Y$ if $E[(X-K)^+] \leq E[(Y-K)^+]$ for all $K \in \mathbb{R}$
  - stop loss order and icx-order are equivalent
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \Leftrightarrow X \leq_{cx} Y$
- $X \leq_{less-dangerous} Y$ if there exists x_0 such that $F_X(x) \leq F_Y(x)$ for all $x \leq x_0$ and $F_X(x) \geq F_Y(x)$ for all $x \geq x_0$ and moreover $E[X] \leq E[Y]$
  - less dangerous order $\Rightarrow$ icx-order or stop loss order

Stochastic orders

Definition (Supermodular function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if for all $x, y \in \boldsymbol{R}^n$

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y) \quad \text{when}$$

$x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$ and
$x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$

- $X \leq_{sm} Y$ if $E[f(X)] \leq E[f(Y)]$ for all supermodular functions f

Review of literature

- Shaked and Shanthikumar(1994)
  Stochastic Orders and Their Applications.
- Müller and Stoyan(2002)
  Comparison Methods for Stochastic Models and Risks.
- Denuit, Dhaene, Goovaerts and Kaas(2005)
  Actuarial Theory for Dependent Risks - Measures, Orders and Models.

Review of literature

- Müller(1997)
  Stop-loss order for portfolios of dependent risks.

$$(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n) \Rightarrow \sum_{i=1}^{n} M_i X_i \leq_{sl} \sum_{i=1}^{n} M_i Y_i$$

- Bäuerle and Müller(2005)
  Stochastic orders ans risk measures: Consistency and bounds

$$X \leq_{icx} Y \Rightarrow \rho(X) \leq \rho(Y)$$

  for all law-invariant, convex risk measures ρ

- Lefèvre and Utev(1996)
  Comparing sums of exchangeable bernoulli random variables.

$$\tilde{p} \leq_{icx} \tilde{p}^* \Rightarrow \sum_{i=1}^{n} D_i \leq_{sl} \sum_{i=1}^{n} D_i^*$$

Supermodular order for Exchangeable Bernoulli random vectors

Theorem

Let $\mathbf{D} = (D_1, \ldots, D_n)$ and $\mathbf{D}^* = (D_1^*, \ldots, D_n^*)$ be two exchangeable Bernoulli random vectors with (resp.) F and F^* as mixture distributions. Then:

$$F \leq_{cx} F^* \quad \Rightarrow \quad \mathbf{D} \leq_{sm} \mathbf{D}^* \quad \text{and}$$
$$F \leq_{icx} F^* \quad \Rightarrow \quad \mathbf{D} \leq_{ism} \mathbf{D}^*$$

Supermodular order for Exchangeable Bernoulli random vectors

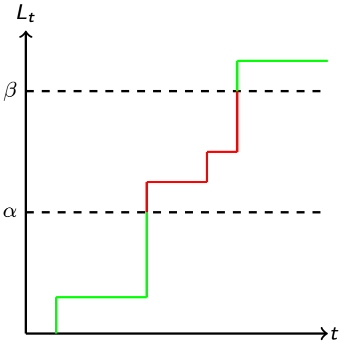
Theorem

Let $D_1, \ldots, D_n, \ldots$ and $D_1^*, \ldots, D_n^*, \ldots$ be two exchangeable sequences of Bernoulli random variables. We denote by F (resp. F^*) the distribution function associated with the mixing measure. Then,

$$(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow F \leq_{cx} F^*. \quad (1)$$

Ordering of CDO tranche premiums

- CDO: Collateralized Debt Obligation
  - insurance contract which covers portfolio losses L_t
  - in a certain tranche $[\alpha, \beta]$ of the total notional



- Tranche premiums only involves call options on the accumulated losses L_t:

$$E[(L_t - \alpha)^+] - E[(L_t - \beta)^+]$$

Ordering of CDO tranche premiums

Burtschell, Gregory, and Laurent(2005a)
A Comparative Analysis of CDO Pricing Models

- Supermodular order for some factor copula models
  - Gaussian copula
  - Student t copula
  - Clayton copula
  - Marshall-Olkin copula

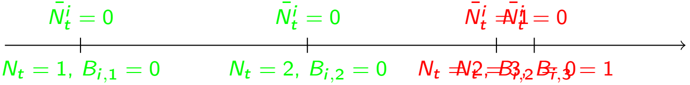
Burtschell, Gregory, and Laurent(2005b)
Beyond the Gaussian Copula: Stochastic and Local Correlation

- Stochastic correlation

Multivariate Poisson model

Duffie(1998), Lindskog and McNeil(2003), Elouerkhaoui(2006)

- $\bar{N}_t^i$ Poisson with parameter $\bar{\lambda}$: idiosyncratic risk
- N_t Poisson with parameter λ: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter p
- All sources of risk are independent
- $N_t^i = \bar{N}_t^i + \sum_{j=1}^{N_t} B_j^i, \ i = 1 \dots n$
- $\tau_i = \inf\{t > 0 | N_t^i > 0\}, \ i = 1 \dots n$



Multivariate Poisson model

- $\tau_i \sim Exp(\bar{\lambda} + p\lambda)$
- Dependence structure of $(\tau_1, \ldots, \tau_n)$ is the Marshall-Olkin copula
- $D_i = 1_{\{\tau_i \leq t\}}, \ i = 1 \ldots n$ are independent knowing N_t
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)$
- Conditional default probability:

$$\tilde{p} = 1 - (1-p)^{N_t} \exp(-\bar{\lambda} t)$$

Multivariate Poisson model

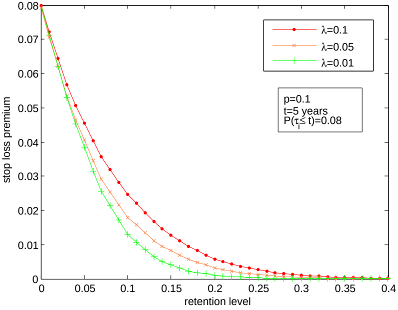
- Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$
- Supermodular order comparison requires equality of marginals: $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^*$
- 3 comparison directions:
  - $p = p^*$: $\bar{\lambda}$ v.s λ
  - $\lambda = \lambda^*$: $\bar{\lambda}$ v.s p
  - $\bar{\lambda} = \bar{\lambda}^*$: λ v.s p

Multivariate Poisson model

Theorem ($p = p^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ *and* $(\bar{\lambda}^*, \lambda^*, p^*)$ *be such that* $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*$, *then:*

$$\lambda \leq \lambda^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



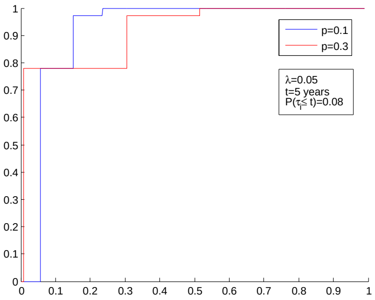
- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- Stop-loss premiums are ordered...

Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar\lambda, \lambda, p)$ *and* $(\bar\lambda^*, \lambda^*, p^*)$ *be such that* $\bar\lambda + p\lambda = \bar\lambda^* + p^*\lambda$, *then:*

$$p \leq p^*,\ \bar\lambda \geq \bar\lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



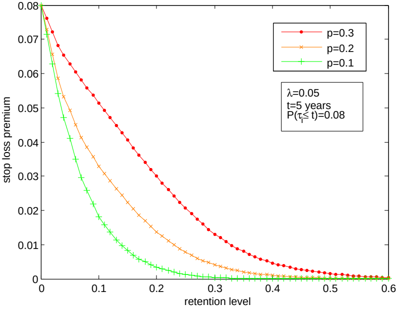
- Less Dangerous order for mixture distributions

Multivariate Poisson model

Theorem ($\lambda = \lambda^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*,\ \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$



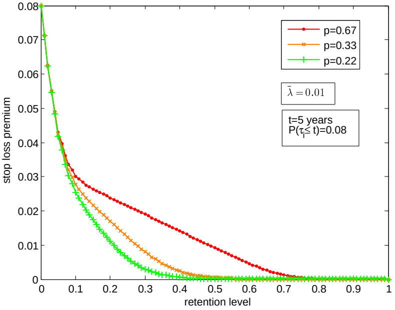
- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1,\ i = 1 \ldots n$
- Stop-loss premiums are ordered...

Multivariate Poisson model

Theorem ($\bar{\lambda} = \bar{\lambda}^*$)

Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $p\lambda = p^*\lambda^*$, then:

$$p \leq p^*, \ \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$



- Computation of $E[(L_t - K)^+]$:
  - 30 names
  - $M_i = 1, \ i = 1 \dots n$
- Stop-loss premiums are ordered...

Structural Model

Hull, Predescu and White(2005)

- Consider n firms
- Let $X_t^i, \ i = 1 \dots n$ be their asset dynamics

$$X_t^i = \sqrt{\rho} W_t + \sqrt{1-\rho} W_t^i, \quad i = 1 \dots n$$

- $W, \ W^i, \ i = 1 \dots n$ are independent standard Wiener processes
- Default times as first passage times:

$$\tau_i = \inf\{t \in \boldsymbol{R}^+ | X_t^i \leq f(t)\}, \quad i = 1 \dots n, \ f : \boldsymbol{R} \to \boldsymbol{R} \text{ continuous}$$

- $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ are independent knowing $\sigma(W_t, \ t \in [0, T])$

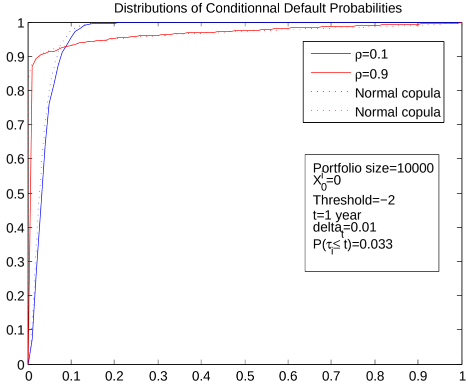
Structural Model

Theorem

For any fixed time horizon T, denote by $D_i = 1_{\{\tau_i \leq T\}}$, $i = 1 \dots n$ and $D_i^* = 1_{\{\tau_i^* \leq T\}}$, $i = 1 \dots n$ the default indicators corresponding to (resp.) ρ and ρ^*, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \dots, D_n) \leq_{sm} (D_1^*, \dots, D_n^*)$$

Structural Model



- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} \tilde{p}$
- $\frac{1}{n}\sum_{i=1}^{n} D_i^* \xrightarrow{a.s} \tilde{p}^*$
- Less Dangerous order for mixture distributions

Archimedean copula

Cossette, Gaillardetz, Marceau and Rioux(2002), Wei and Hu(2002)

- V is a positive random variable with Laplace transform φ^{-1}
- $U_1, \ldots, U_n$ are independent Uniform random variables independent of V
- $V_i = \varphi^{-1}\left(-\frac{\ln U_i}{V}\right)$, $i = 1 \ldots n$
  - $(V_1, \ldots, V_n)$ follows a φ-archimedean copula
  - $P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}(\varphi(v_1) + \ldots + \varphi(v_n))$
- $\tau_i = G^{-1}(V_i)$
  - G: distribution function of τ_i
- $D_i = 1_{\{\tau_i \leq t\}}$, $i = 1 \ldots n$ independent knowing V
- $\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \leq t \mid V)$
- Conditional default probability:

$$\tilde{p} = \exp\{-\varphi(G(t)V)\}$$

Archimedean copula

Copula name	Generator φ	V-distribution
Clayton	$t^{-\theta} - 1$	Gamma$(1/\theta)$
Gumbel	$(-\ln(t))^{\theta}$	α-Stable, $\alpha = 1/\theta$
Franck	$-\ln\left[(1-e^{-\theta t})/(1-e^{-\theta})\right]$	Logarithmic series

Theorem

$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

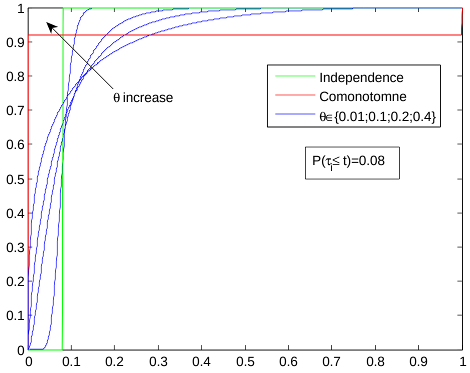
- The proof derived from the following result:

Theorem

Let V and V^* be two positive random variables. Denote by φ^{-1} et ψ^{-1} their Laplace transform.
Consider $\tilde{p} = \exp(-\varphi(G(t))V)$ and $\tilde{p}^* = \exp(-\psi(G(t))V^*)$, the corresponding conditional default probabilities, then:

$$\varphi \circ \psi^{-1} \in \mathscr{L}_\infty^* = \{f : \mathbf{R}^+ \to \mathbf{R}^+ | (-1)^{n-1} f^{(n)} \geq 0 \; \forall n \geq 1\} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*$$

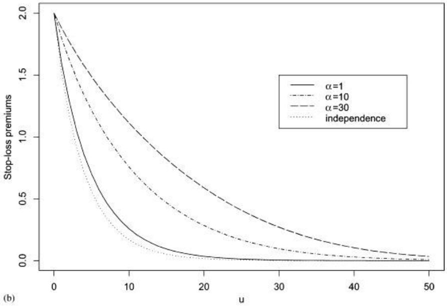
Archimedean copula



- Clayton copula
- Less Dangerous order for mixture distributions

Archimedean copula

- Previous result is consistent with
  - Cossette, Gaillardetz, Marceau and Rioux(2002)
- Computation of $E[(L_t - u)^+]$ with $L_t = \sum_{i=1}^{20} M_i D_i$
- $(D_1, \ldots, D_{20})$ follows a Clayton copula with parameter α
- $P(D_i = 1) = 0.05$
- $M_i \sim \text{Gamma}(1, 2)$



Double t copula

Hull and White(2004)

- $V \sim t(\nu)$, $\bar{V}_i \sim t(\bar{\nu})$ Student with ν (resp.) $\bar{\nu}$ degree of freedom

$$V_i = \rho \left(\frac{\nu - 2}{\nu}\right)^{1/2} V + \sqrt{1-\rho^2} \left(\frac{\bar{\nu} - 2}{\bar{\nu}}\right)^{1/2} \bar{V}_i$$

- $\tau_i = G^{-1}(H_\rho(V_i))$, $i = 1 \dots n$
  - G: distribution function of τ_i
  - H_ρ: distribution function of V_i
- $D_i = 1_{\{\tau_i \le t\}}$, $i = 1 \dots n$ independent knowing V
- $\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s} E[D_i \mid V] = P(\tau_i \le t \mid V)$
- Conditional default probability:

$$\tilde{p} = t_{\bar{\nu}} \left(\left(\frac{\bar{\nu}}{\bar{\nu} - 2}\right)^{1/2} \frac{H_\rho^{-1}(G(t)) - \rho \left(\frac{\nu-2}{\nu}\right)^{1/2} V}{\sqrt{1-\rho^2}} \right)$$

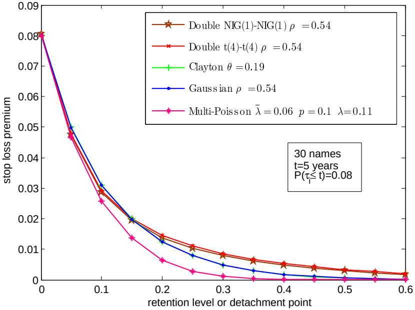
Double t copula

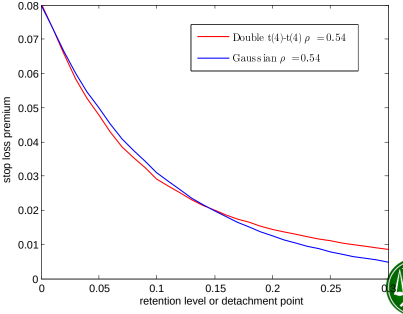
Theorem

$\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$

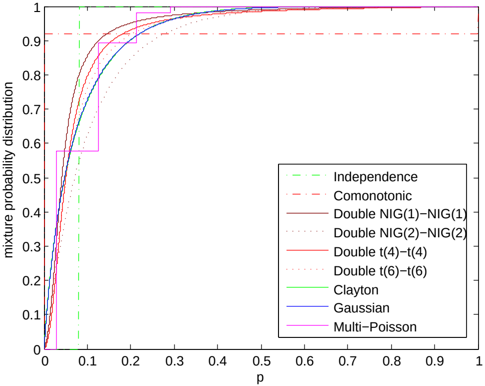
Comparison of different models

- Stop loss premium: $E[(L_t - K)^+]$ with $L_t = \frac{1}{n}\sum_{i=1}^{n} D_i$
- Comparison criteria:
  - same default marginals for all models
  - dependence parameters set to get equal premiums for $K = 0.03$

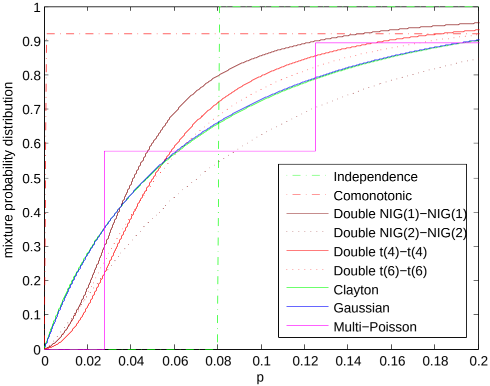




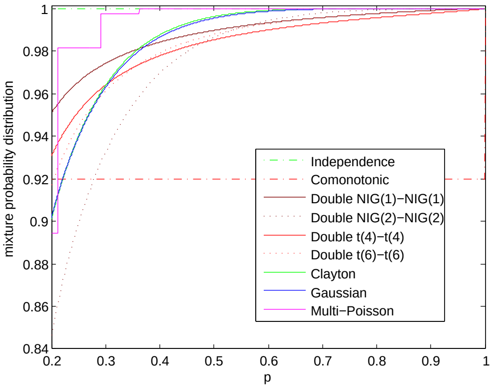
Comparison of different models



Comparison of different models



Comparison of different models



Conclusion

- Characterization of supermodular order for exchangeable Bernoulli random vectors
- Comparison of CDO tranche premiums or reinsurance premiums in the individual life model
- Unified way of presenting default risk models...