On Multivariate Extensions of Value-at-Risk

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Joint work with Elena Di Bernardino, CNAM

AFFI Spring International Meeting

Lyon, May 28, 2013

- A. Cousin, E. Di Bernardino, *On Multivariate Extensions of Value-at-Risk*, submitted to *Journal of Multivariate Analysis*
  Available on HAL: http://hal.archives-ouvertes.fr/hal-00638382

- A. Cousin, E. Di Bernardino, *On Multivariate Extensions of Conditional-Tail-Expectation*, in preparation

- Regulatory capital rule relies on the VaR paradigm and the risk diversification effect.
- How could we deal with risks that cannot be aggregated together ?
- Presence of non-monetary risks ?
- Exogenous risks ?

Construction of Multivariate Risk Measures

$$\rho : \quad \mathbf{X} := (X_1, \ldots, X_d) \mapsto \begin{pmatrix} \rho^1[\mathbf{X}] \\ \vdots \\ \rho^d[\mathbf{X}] \end{pmatrix} \in \mathbb{R}_+^d,$$

Some desirable properties:

- Combine in a concise way information on both marginals and risks dependencies
- Compatible with univariate version when $d = 1$
- Easily computable for large class of multivariate distribution functions
- Consistent with usual invariance properties (Artzner et al.'s axioms)
- Consistent behavior with respect to risk perturbations

Multivariate *Value-at-Risk* as quantile curve (Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009):

$$\partial \underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) = \alpha\} \qquad \partial \overline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : \overline{F}(\mathbf{x}) = 1 - \alpha\}$$

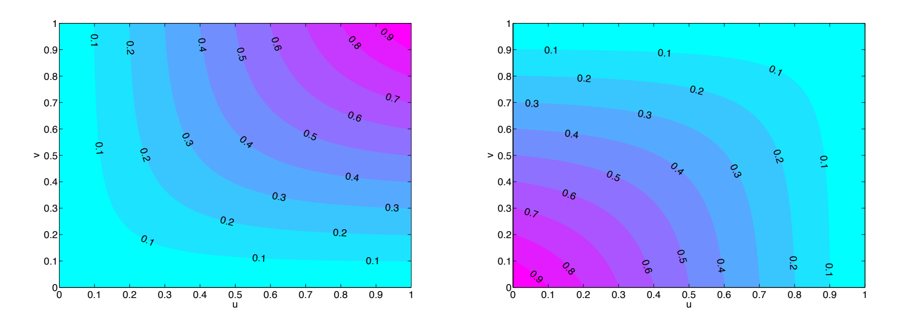

Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Value-at-Risk**

Definition

Consider a random vector $\mathbf{X}$ with absolutely continuous cdf F and survival function $\overline{F}$. For $\alpha \in (0,1)$, we define:

$$\underline{\mathrm{VaR}}_\alpha(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) = \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) = \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) = \alpha \,] \end{pmatrix}$$

$$\overline{\mathrm{VaR}}_\alpha(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \overline{F}(\mathbf{X}) = 1 - \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \overline{F}(\mathbf{X}) = 1 - \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \overline{F}(\mathbf{X}) = 1 - \alpha \,] \end{pmatrix}$$

When $d = 1$: $\underline{\mathrm{VaR}}_\alpha(X) = \overline{\mathrm{VaR}}_\alpha(X) = \mathrm{VaR}_\alpha(X)$

VaRs for risk portfolios with Archimedean copula dependence structure

Proposition

Let $\mathbf{X}$ *be a d-dimensional portfolio of risks with marginal distributions* $F_1, \dots, F_d$.

- *If* $\mathbf{X}$ *admits an Archimedian copula with generator* ϕ*, then*

$$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = \mathbb{E}\left[F_i^{-1}\left(\phi^{-1}(S_i\phi(\alpha))\right)\right], \quad i = 1, \dots, d$$

- *If* $\tilde{\mathbf{X}}$ *admits an Archimedian survival copula with generator* ϕ*, then*

$$\overline{\mathrm{VaR}}^i_\alpha(\tilde{\mathbf{X}}) = \mathbb{E}\left[F_i^{-1}\left(1 - \phi^{-1}(S_i\phi(1-\alpha))\right)\right], \quad i = 1, \dots, d$$

where S_i *is a random variable with* $\mathrm{Beta}(1, d-1)$ *distribution.*

Explicit expressions for bivariate Clayton copulas

Copula	θ	$\underline{\mathrm{VaR}}^{i}_{\alpha,\theta}(X,Y)$	$\overline{\mathrm{VaR}}^{i}_{\alpha,\theta}(\tilde{X},\tilde{Y})$
Clayton C_θ	$(-1,\infty)$	$\frac{\theta}{\theta-1}\frac{\alpha^\theta-\alpha}{\alpha^\theta-1}$	$1-\frac{\theta}{\theta-1}\frac{(1-\alpha)^\theta-(1-\alpha)}{(1-\alpha)^\theta-1}$
Counter-monotonic	-1	$\frac{1+\alpha}{2}$	$\frac{\alpha}{2}$
Independent	0	$\frac{\alpha-1}{\ln\alpha}$	$1+\frac{\alpha}{\ln(1-\alpha)}$
Comonotonic	∞	α	α

Table: Components $i=1,2$ of $\underline{\mathrm{VaR}}^{i}$ and $\overline{\mathrm{VaR}}^{i}$ where (X,Y) follows a Clayton copula and $(\tilde{X},\tilde{Y}) := (1-X, 1-Y)$, i.e., $(\tilde{X},\tilde{Y})$ has a survival Clayton copula with uniform margins

Behavior of VaR components: bivariate Clayton case

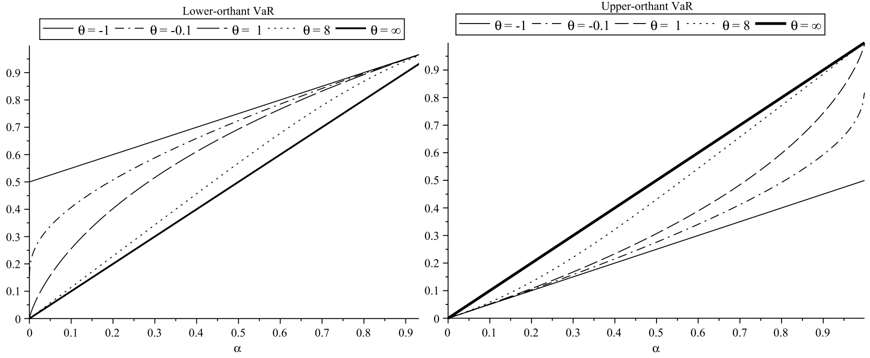

Figure: Behavior of $\underline{\mathrm{VaR}}^{1}_{\alpha,\theta}(X, Y)$ (**left**) and $\overline{\mathrm{VaR}}^{1}_{\alpha,\theta}(\tilde{X}, \tilde{Y})$ (**right**) with respect to risk level α and dependence parameter θ

Invariance properties, comparison with univariate VaR, behavior with respect to α

	$\underline{\mathrm{VaR}}_\alpha(\mathbf{X})$	$\overline{\mathrm{VaR}}_\alpha(\mathbf{X})$
Several (axiomatic) properties	Invariance properties ($\mathbf{c} \in \mathbb{R}^d_+$): • $\underline{\mathrm{VaR}}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\underline{\mathrm{VaR}}_\alpha(\mathbf{X})$, • $\underline{\mathrm{VaR}}_\alpha(\mathbf{c}+\mathbf{X}) = \mathbf{c}+\underline{\mathrm{VaR}}_\alpha(\mathbf{X})$. Univariate VaR is a lower bound: • $\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) \geq \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. Comonotonic case: • $\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$.	Invariance properties ($\mathbf{c} \in \mathbb{R}^d_+$): • $\overline{\mathrm{VaR}}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\overline{\mathrm{VaR}}_\alpha(\mathbf{X})$, • $\overline{\mathrm{VaR}}_\alpha(\mathbf{c}+\mathbf{X}) = \mathbf{c}+\overline{\mathrm{VaR}}_\alpha(\mathbf{X})$. Univariate VaR is an upper bound: • $\overline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) \leq \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. Comonotonic case: • $\overline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$.
Risk level	$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α. ($\mathbf{X}$ with Archimedean copulas)	$\overline{\mathrm{VaR}}^i_\alpha(\tilde{\mathbf{X}})$ is a non-decreasing function of α. ($\tilde{\mathbf{X}}$ with Archimedean survival copulas)

Effect of a risk perturbation

	$\underline{\mathrm{VaR}}_\alpha(\mathbf{X})$	$\overline{\mathrm{VaR}}_\alpha(\mathbf{X})$
Change in marginals	For a fixed copula C, if $X_i \overset{d}{=} Y_i$: • $\underline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{X}) = \underline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. For a fixed copula C, if $X_i \preceq_{st} Y_i$: • $\underline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{X}) \le \underline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$.	For a fixed copula C, if $X_i \overset{d}{=} Y_i$: • $\overline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{X}) = \overline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. For a fixed copula C, if $X_i \preceq_{st} Y_i$: • $\overline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{X}) \le \overline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$.
Change in dependence structure	For fixed marginals, if $\theta \le \theta^*$: • $\underline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{X}) \le \underline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. $\mathbf{X}$ with Archimedean copula	For fixed marginals, if $\theta \le \theta^*$: • $\overline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{X}) \le \overline{\mathrm{VaR}}^{\boldsymbol{i}}_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. $\tilde{\mathbf{X}}$ with Archimedean survival copula

Perspectives

- Are efficient computation procedure available for other kind of dependence structures (heterogeneous one in particular) ?
- What about Multivariate CTE ?
- Comparisons of our multivariate VaR and CTE with existing multivariate risk measures
- Extension to discrete distribution functions

Thank you for your attention

Other multivariate risk measures in the literature

Several multivariate generalizations of CTE. For $i = 1, \ldots, d$

- $\mathrm{CTE}_{\alpha}^{\mathrm{sum}}(X_i) = \mathbb{E}[\, X_i \,|\, S > Q_S(\alpha) \,]$ where $S = X_1 + \cdots + X_d$
- $\mathrm{CTE}_{\alpha}^{\mathrm{min}}(X_i) = \mathbb{E}[\, X_i \,|\, X_{(1)} > Q_{X_{(1)}}(\alpha) \,]$ where $X_{(1)} = \min\{X_1, \ldots, X_d\}$
- $\mathrm{CTE}_{\alpha}^{\mathrm{max}}(X_i) = \mathbb{E}[\, X_i \,|\, X_{(d)} > Q_{X_{(d)}}(\alpha) \,]$ where $X_{(d)} = \max\{X_1, \ldots, X_d\}$

For Farlie-Gumbel-Morgenstern copula (Bargès *et al.*, 2009). For elliptic distribution functions (Landsman and Valdez, 2003). For phase-type distributions (Cai and Li, 2005).

- Inappropriate to measure risks with heterogeneous characteristics especially in an external risks problem

Multivariate *CTE*-s based on upper-level set of multivariate cdf and lower-level set of survival functions:

$$\underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}^d_+ : F(\mathbf{x}) \geq \alpha\} \qquad \overline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}^d_+ : \overline{F}(\mathbf{x}) \leq 1 - \alpha\}$$

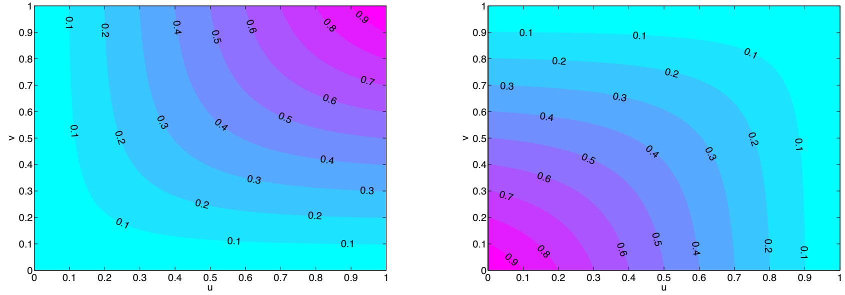

Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Conditional-Tail-Expectation**

Definition

Consider a random vector $\mathbf{X}$ with absolutely continuous cdf F and survival function $\overline{F}$. For $\alpha \in (0,1)$, we define:

$$\underline{\mathrm{CTE}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) \geq \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) \geq \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) \geq \alpha \,] \end{pmatrix}$$

$$\overline{\mathrm{CTE}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \overline{F}(\mathbf{X}) \leq 1-\alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \overline{F}(\mathbf{X}) \leq 1-\alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \overline{F}(\mathbf{X}) \leq 1-\alpha \,] \end{pmatrix}$$

CTE explicit expressions for bivariate Clayton copulas

Copula	θ	$\underline{\mathrm{CTE}}^{i}_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1,\infty)$	$\frac{1}{2}\,\frac{\theta}{\theta-1}\,\frac{\theta-1-\alpha^2(1+\theta)+2\,\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic W	-1	$\frac{1}{4}\,\frac{1-\alpha^2+2\ln\alpha}{1-\alpha+\ln\alpha}$
Independent Π	0	$\frac{1}{2}\,\frac{(1-\alpha)^2}{1-\alpha+\alpha\ln\alpha}$
Comonotonic M	∞	$\frac{1+\alpha}{2}$

Table: Components $i=1,2$ of $\underline{\mathrm{CTE}}^{i}$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \underline{\mathrm{CTE}}^{i}_{\alpha,\theta}}{\partial \alpha} \geq 0$ and $\frac{\partial \underline{\mathrm{CTE}}^{i}_{\alpha,\theta}}{\partial \theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0,1)$.

Behavior of CTE components: bivariate Frank case

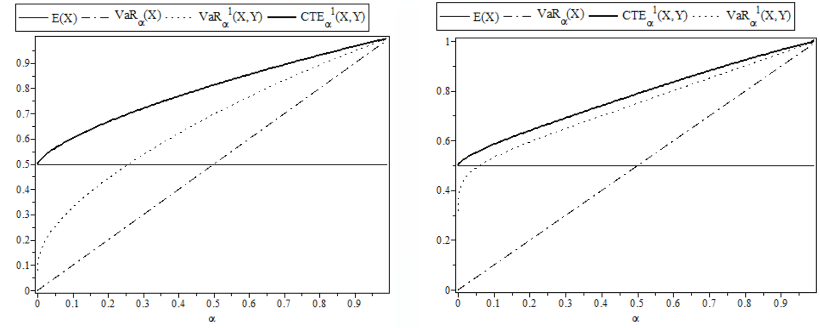

Figure: Frank copula with standard uniform marginals, parameter $\theta = 2$ (left), parameter $\theta = -10$ (right).