Chapter 1
Valuation of portfolio loss derivatives in an infectious model*

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Abstract
In this paper we investigate a particular specification of the top-down dynamic contagion model proposed in "An extension of Davis and Lo's contagion model (2010)". We consider an economy of n firms which may default directly or may be infected by other defaulting firms (a domino effect being also possible). The spontaneous default without external influence and the infections are described by conditionally independent Bernoulli-type random variables. We provide a recursive algorithm for the computation of the loss distribution that involves successive applications of the so-called Waring's formula. The major advantage of this algorithm is that it can be applied for a large portfolio. We then examine the calibration of model parameters on CDX.NA.IG tranche quotes during the crisis.

Key words: credit risk, contagion model, dependent defaults, default distribution, exchangeability, CDO tranches

1.1 Introduction

The recent financial crisis marked the need for paying more attention to the systemic risk which can partially be the result of dependence on many factors to a global economic environment. A tractable and common way of modeling dependence among default events is to rely on the conditional independence assumption. Conditionally on the evolution of some business cycle or macroeconomic related factors, defaults are assumed to be independent. However, as shown by some empirical studies such

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as [7, 12] or [2], the latter assumption seems to be rejected when tested on historical default data. An additional source of dependence, namely the chain contagion effect, is observed and requires the construction of contagion models which would be able to explain the "domino effects": a defaulting firm causes the default of another firm which infects another one etc.
In this paper we consider a particular model inspired from [4] which will be shortly summarized here. In this previous extension, we studied the case of credit entities that can default either directly or by infection. We extended Davis and Lo's framework, by relaxing the iid assumption of direct defaults and the iid assumption of contaminations. We also introduced some features allowing to take into account a higher number of contaminations required to cause a direct default. Furthermore, the one-period setting in Davis and Lo's paper was extended to a fully dynamic discrete-time setting. Compared to Davis and Lo's model in which only directly defaulting bonds can infect others, our model accounts for a “domino effect” which can exist between firms due to counterparty relationships. Thus in the model presented here, the firms can default because of a chain reaction, phenomena which is often a reason for financial crises. However, in this model, there is no inter-temporal effect in the infection [see 3, for a model in which there is a delayed effect between defaults and contagion].
The model proposed in this paper preserves the exchangeability assumption of the previous model, but is more specific, in order to reduce the complexity of several formulas and to cope with numerical instability of some of our previous results. This model is based on conditional independence assumption. Particularly, direct defaults and contaminations are assumed to be mixtures of independent Bernoulli variables mixed with a Beta-distributed factor. The main contribution of the paper is a tractable expression for the distribution of the total number of defaults. The latter expression can be computed by successive application of the same analytical function based on the so-called Waring's formula. This is very appealing on practical grounds, given that the latter formula can be computed efficiently using recursive algorithms [see 5].
The outline of the present paper is as follows : in Section 1.2 we present the Davis and Lo's model and a previous extension of their model. Then, in Section 1.3, we analyze a particular case of this extension and give a specific algorithm more suitable for large portfolios case. At last, in Section 1.4, we present a short numerical application of this model to CDX.NA.IG tranche quotes during the crisis.

1.2 Previous studies

Davis and Lo have presented a model where each credit reference can default either directly, or may be infected by other defaulted references. Let n be the number of credit references. For name i, we denote by X_i the direct default indicator, $\mathscr{C}_i$ the indirect default indicator and Z_i the default indicator (direct or indirect). The Davis and Lo's one-period model, may be written as follows: $Z_i = X_i + (1 - X_i)\mathscr{C}_i$.

Default of name i occurs if there is a direct default $X_i = 1$, or otherwise if there is a contamination $X_i = 0$ and $\mathscr{C}_i = 1$. The contagion occurs if at least another reference j defaults directly ($X_j = 1$), and contaminates the considered reference i ($Y_{ji} = 1$), so that: $\mathscr{C}_i = \mathbb{1}_{\text{at least one } X_j Y_{ji}=1,\ j=1,\dots,n} = \mathbb{1}_{\sum_{j=1\dots n, j\neq i} X_j Y_{ji} \geq 1}$.

Davis and Lo obtain the distribution of the total number of defaults $N = \sum_{i=1}^n Z_i$:

$$\mathbb{P}[N=k] = \mathrm{C}_n^k \sum_{i=1}^{k} \mathrm{C}_k^i p^i (1-p)^{n-i} (1-(1-q)^i)^{k-i} (1-q)^{i(n-k)}.$$

where $\mathrm{C}_n^k = \frac{n!}{k!(n-k)!}$. This result is obtained under the following assumptions:

- $\{X_i,\ i=1,\dots,n\}$ are iid Bernoulli r.v. with parameter p,
- $\{Y_{ij},\ i,j=1,\dots,n\}$ are iid Bernoulli r.v. with parameter q,
- At least one infection causes an indirect default,
- An infected entity cannot contaminate others (no chain-reaction effect).

We showed in a previous paper [see 4] that these assumptions were quite restrictive, so that it is important to release them. One of the most important feature of this paper is to consider a contagion credit risk model with several periods $[t,\ t+1]$, $t \in \{1,\dots,T\}$, where $T \in \mathbb{N}^*$ is the maximum time horizon.

Recall that n is the number of names in the credit portfolio and $\Omega = \{1,\dots,n\}$ the corresponding set of entities. We denote by X_t^i the direct default indicator, $\mathscr{C}_t^i$ the indirect default indicator, Z_t^i the default indicator (direct or indirect) associated with name i in the period $[t,\ t+1[$. The model is:

$$\begin{cases} Z_0^i = 0,\ i=1,\dots,n \\ Z_t^i = Z_{t-1}^i + (1-Z_{t-1}^i)[X_t^i + (1-X_t^i)\mathscr{C}_t^i],\ i=1,\dots,n,\ t=1,\dots,T, \end{cases} \tag{1.1}$$

where $\mathscr{C}_t^i = f\left(\sum_{j\in F_t} Y_t^{ji}\right)$ and

- Y_t^{ji}, $i,j=1,\dots,n$ are Bernoulli random variables such that $Y_t^{ji} = 1$ if entity j infects entity i between t and $t+1$,
- F_t is the set of the defaulting entities that are likely to infect other entities between t and $t+1$. Here, F_t is the set of entities that have defaulted directly during this period [like in 8]. Other choices allow inter-periodic contagion effect [see 4, 3].
- f is a contamination trigger function, for example $f(x) = \mathbb{1}_{x\geq 1}$ (like in Davis and Lo's model) or $f(x) = \mathbb{1}_{x\geq 2}$ (several infections may be required to cause an indirect default, two in this particular case).

Hence, $Z_t^i = 1$ if the entity has been declared in default at the end of period $t-1$ ($Z_{t-1}^i = 1$) or if, during the period $[t,\ t+1]$, it defaults directly ($X_t^i = 1$) or by infection ($\mathscr{C}_t^i = 1$).

Again, each credit entity can default either directly or by infection of other references. Nevertheless, two features have been extended: the monoperiodic framework is changed into a multiperiodic framework, and the contamination trigger function is more general than Davis and Lo's one. From now on, the following notations are used throughout the paper:

Notation 1 *For every $t \in \{1,\dots,T\}$, we denote by:*

- *Γ_t the set of entities which did not default in the previous periods: $\Gamma_t = \left\{i \in \Omega, Z_t^i = 0\right\}$,*
- *N_t^D (resp. N_t^C) the number of direct (resp. indirect) defaults during the period $[t, t+1[$: $N_t^D = \sum_{i \in \Gamma_{t-1}} X_t^i$ (resp. $N_t^C = \sum_{i \in \Gamma_{t-1}} (1 - X_t^i)\mathscr{C}_t^i$),*
- *N_t the number of defaults occurred up to time t: $N_t = \sum_{i \in \Omega} Z_t^i = N_{t-1} + N_t^D + N_t^C$,*
- *N_t^R the residual number of non defaulted entities at time t, $N_t^R = n - N_t$.*

The aim is to study the law of N_t under the following assumption :

Assumption 1 (Direct defaults and contamination sequence)

- *The random vectors $\overrightarrow{X_t} = (X_t^1, ..., X_t^n)$, $t \in \{1,\dots,T\}$, are mutually independent, but their components are exchangeable.*
- *The vectors $\overrightarrow{Y_t} = (Y_t^{11}, Y_t^{12}, ..., Y_t^{nn})$, $t \in \{1,\dots,T\}$, are mutually independent.For all $t \in \{1,\dots,T\}$, the variables $\left\{Y_t^{ji},\ (j,i) \in \Omega^2\right\}$ are exchangeable (and independent of $\left\{X_t^i,\ t = 1,\dots,T,\ i \in \Omega\right\}$).*

Theorem 2 (Distribution of the N_t, exchangeable case, T periods). *Under Assumption 1, the distribution of N_t is given by the recursive formula:*

$$\begin{cases} \mathbb{P}[N_0 = r] = \mathbb{1}_{r=0}, \ r=0,\dots,n \\ \mathbb{P}[N_t = r] = \sum_{k=0}^{r} \mathbb{P}[N_t = r | N_{t-1} = k]\mathbb{P}[N_{t-1} = k], \ r=0,\dots,n \end{cases} \tag{1.2}$$

$$\mathbb{P}[N_t = r | N_{t-1} = k] = C_{n-k}^{r-k} \sum_{\gamma=0}^{r-k} C_{r-k}^{\gamma} \sum_{\alpha=0}^{n-k-\gamma} C_{n-k-\gamma}^{\alpha} \mu_{\gamma+\alpha,t} \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^{j+\alpha} \xi_{j+r-k-\gamma,t}(\gamma),$$

$$\text{and} \begin{cases} \mu_{k,t} = \mathbb{P}\left[X_t^1 = 1 \cap \dots \cap X_t^k = 1\right], \ 1 \le k \le n, \\ \xi_{k,t}(\gamma) = \mathbb{P}\left[\mathscr{C}_t^1 = 1 \cap \dots \cap \mathscr{C}_t^k = 1 \middle| N_t^D = \gamma\right], \ 1 \le k \le n-\gamma, \ \gamma \le n, \\ \xi_{0,t}(\gamma) = 1 \text{ (including the case } \gamma = 0). \end{cases} \tag{1.3}$$

The coefficients $\xi_{k,t}(\cdot)$ may be computed recursively. For more details and proof of this theorem, see [4]. This formula was obtained using life insurance tools (namely Waring's formula). If the number of underlying credit entities is too large, some difficulties may arise when it turns to compute such a formula:

- First, expression (1.2) involves three successive sums and may lead to large computation time. When n is large, one needs to pre-compute parts of these sums to fasten the computation. As a consequence, this expression should be transformed in order to reduce time complexity.
- Second, this can lead to very large binomial coefficients and to numerical issues due to the limited floating point precision of the computer.

It can be useful to get special cases of the model leading to straightforward computations and to a greater stability of the formula.

1.3 The model

As in the original Davis and Lo's paper, the contamination trigger function is equal to $f(x) = \mathbb{1}_{x\geq 1}$ and the set F_t of entities likely to contaminate others is here the set of entities which default directly on period t. We examine here how formulas in Theorem 2 can be clarified under conditional independence assumption:
-The components of $\overrightarrow{X_t}$ are conditionally independent given a random variable Θ_X,
-The components of $\overrightarrow{Y_t}$ are conditionally independent given a random variable Θ_Y.
Consider m indicator random variables $X_1,\ldots,X_m \in \{0,1\}$. Suppose that these random variables are mixtures of mutually iid Bernoulli random variables. In other words, $X_1,\ldots,X_m$ are conditionally independent Bernoulli's with a common random parameter Θ. More precisely, for any $i \in \Omega$, the probability that X_i equals one is thus given by the latent factor Θ. This corresponds to the situation where each probability is governed by a common macro-economic environment variable Θ:

$$\mathbb{P}[X_1+\cdots+X_m = k] = \int_0^1 C_m^k \theta^k (1-\theta)^{m-k} dF_\Theta(\theta)\,, \tag{1.4}$$

where F_Θ denotes the distribution function of Θ. Let us note that the distribution of the sum $X_1+\cdots+X_m$ is a Binomial mixture. Of course, numerical integration techniques may be used to compute expression (1.4). But, as described below, exact quantities can be extracted when the moments of the underlying factor Θ are known. To this aim we use Waring's formula, which is well known in the actuarial field, see [9] or [10] for an older reference. Remark that, with an underlying random factor Θ, $\mathbb{P}[X_1 = 1,\ldots,X_j = 1] = \mathbb{E}[\mathbb{P}[X_1 = 1,\ldots,X_j = 1|\Theta]] = \mathbb{E}\left[\Theta^j\right]$, for $j \in \{1,\ldots m\}$.

Theorem 3 (Waring's formula, Binomial mixture). *If $\mu_k = \mathbb{E}\left[\Theta^k\right]$ is the k^{th} moment of the underlying factor Θ, then for $k \in \{0,\ldots,m\}$:*

$$\mathbb{P}\left[\sum_{i=1}^m X_i = k\right] = \mathbb{1}_{k\leq m} C_m^k \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \mu_{j+k}\,.$$

Proof: Some elements of the proof and references are given in [4], and in [9]. □

We will see that it is interesting to express Waring's formula as a function of an input vector $\overrightarrow{\nu} = (\nu_1,\ldots,\nu_n)$. Particularly, for $k,z,m \in \mathbb{N}$, $m \leq n$, Waring's formula can be written:

$$W:\ \left(\overrightarrow{\nu},k,m\right) \mapsto\ \mathbb{1}_{k\leq m} C_m^k \textstyle\sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \nu_{j+k}\,. \tag{1.5}$$

Some recursive algorithms for calculating Waring's formula (1.5) are given in [5]. It turns out that these algorithms may significantly improve the computation time. In the special case where Θ has a Beta distribution with parameters α and β, the function W can be expressed as a function of α and β instead of $\overrightarrow{\nu}$: $W(\overrightarrow{\nu},k,m) = \tilde{W}_{\alpha,\beta}(k,m) := \mathrm{C}_m^k \frac{B(\alpha+k,m-k+\beta)}{B(\alpha,\beta)}$ where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the Beta function.

Now consider again the credit risk model described by (1.1). One contribution of the paper is to express the law of number of defaults in terms of successive evaluation of Waring's formula (1.5), the latter formula being applied with different vectors $\overrightarrow{v}$. This allows to clarify all quantities that can be pre-computed, and to reduce the complexity of transition probabilities given by expression (1.2) in Theorem 2. This helps to find which quantity can be solved analytically in some particular cases.

Theorem 4 (Transition probabilities). *With underlying random factors Θ_X and Θ_Y, the transition probabilities of the total number of defaults is given by :*

$$\mathbb{P}[N_t = r | N_{t-1} = k] = \sum_{\gamma=0}^{r-k} p_t^D(\gamma, n-k) p_t^C(r-k-\gamma, \gamma, n-k-\gamma),$$

$$\text{with } \begin{cases} p_t^D(k,m) = \mathbb{P}\left[N_t^D = k \middle| N_t^R = m\right] = W(\overrightarrow{\mu}, k, m), \\ p_t^C(k,z,m) = \mathbb{P}\left[N_t^C = k \middle| N_t^D = z, N_t^R - N_t^D = m\right] = W(\overrightarrow{\xi(z)}, k, m), \end{cases}$$

and where $\mu_k = \mathbb{E}\left[\Theta_X^k\right]$, $\xi_k(z) = W(\overrightarrow{h(z)}, 0, k)$, $h_i(z) = W(\overrightarrow{\lambda}, 0, iz)$, $\lambda_k = \mathbb{E}\left[\Theta_Y^k\right]$ are the components of (resp.) vectors $\overrightarrow{\mu}$, $\overrightarrow{\xi(z)}$, $\overrightarrow{h(z)}$, $\overrightarrow{\lambda}$.

Corollary 1 (Transition probabilities, Beta-dirac case). *If Θ_X is Beta-distributed with parameters (α, β) and $\Theta_Y = q \in [0,\ 1]$ (so that Y_t^{ji} are i.i.d.), then*

$$\begin{aligned} \mathbb{P}[N_t = r | N_{t-1} = k] = \sum_{\gamma=0}^{r-k} & \mathbb{P}\left[N_t^D = \gamma \middle| N_t^R = n-k\right] \\ & \times \mathbb{P}\left[N_t^C = r-k-\gamma \middle| N_t^D = \gamma, N_t^R - N_t^D = n-k-\gamma\right] \end{aligned}$$

with $\mathbb{P}\left[N_t^D = k \middle| N_t^R = m\right] = \tilde{W}_{\alpha,\beta}(k,m)$, $\mathbb{P}\left[N_t^C = k \middle| N_t^D = z, N_t^R - N_t^D = m\right] = \mathbb{1}_{k \leq m} \mathrm{C}_m^k (x_z)^k (1-x_z)^{m-k}$ and $x_z = (1-(1-q)^z)$.

1.4 Calibration of parameters on liquid CDO tranche quotes

In [4], we perform a calibration analysis of model parameters on iTraxx Europe tranches. Here instead, the focus is put on standardized tranches referencing the 5-years CDX North American Investment Grade index (CDX.NA.IG henceforth). The model used for calibration of CDO tranche quotes is such that, for any $t \geq 0$, direct defaults X_t^i, $i = 1, \ldots, n$ are Bernoulli mixtures with a common random parameter that is Beta-distributed with mean p and variance σ^2. The infectious transition links Y_t^{ij}, $1 \leq i, j \leq n$ are independent Bernoulli random variables with the same constant mean q. We also consider the case where only one contamination is required to trigger a default. This corresponds to the assumptions of Corollary 1. Using the latter restrictions, the discrete-time contagion model is stationary and it can be entirely described by the vector of annual scaled parameters $\eta = (p, \sigma, q)$. Note that there is a one-to-one correspondence between parameters (α, β) associated with the Beta-distributed variable Θ_X in Corollary 1 and its mean and standard deviation (p, σ). Let us recall that the computation of CDO tranche spreads only involves the expec-

	0%-3%	3%-7%	7%-10%	10%-15%	15%-30%	index	RMSE	p^*	σ^*	q^*
Market quotes	55	619	321	204	95	143	-	-	-	-
Calibration 1	30	689	406	240	72	91	0.29	0.0074	0.0133	0.010
Calibration 2	-	568	364	237	90	84	0.21	0.0070	0.0154	0.010
Calibration 3	-	540	335	227	88	-	0.09	0.0011	0.0026	0.094
Calibration 4	55	-	-	-	-	143	0	0.0020	0.0002	0.089

Table 1.1 Market and model spreads (in bp) in the four calibrations and the corresponding root mean square errors. The [0%-3%] spread is quoted in %.

tation of tranche losses at several time horizons (see [6] for more details regarding cash-flows of synthetic CDO tranches). In the case where recovery rates are the same across names and equal to a constant R, it is straightforward to remark that the current cumulative loss is merely proportional to the current number of defaults. Then, Theorem 2 and Corollary 1 can be used properly to compute CDO tranche spreads. Let us denote by $\tilde{s}_0$, $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{s}_4$, $\tilde{s}_5$, $\tilde{s}_6$ the market spreads associated with (respectively) the CDS index, [0%-3%], [3%-7%], [7%-10%], [10%-15%] and [15%-30%] standard CDX.NA.IG tranches and by $s_0(\eta)$, $s_1(\eta)$, $s_2(\eta)$, $s_4(\eta)$, $s_5(\eta)$, $s_6(\eta)$, the corresponding spreads generated by the contagion model using the vector of parameters η. The calibration process aims at finding out the optimal parameter set $\eta^* = (p^*, \sigma^*, q^*)$ which minimizes the following least-square objective function $RMSE(\eta)^2 = \frac{1}{6}\sum_{i=1}^{6}(\tilde{s}_i - s_i(\eta))^2/\tilde{s}_i^2$. For both data sets, in order to analyze the calibration efficiency in a deeper way, we have compared the global calibration with three alternative ones, where some of the available market spreads were excluded from the fitting. Here are the calibration procedures we have considered:
-C1: All available market spreads are included in the fitting,
-C2: The equity [0%-3%] tranche spread is excluded,
-C3: Both equity [0%-3%] tranche and CDS index spreads are excluded,
-C4: All tranche spreads are excluded except equity tranche and CDS index spreads.
In all calibrations the interest rate is set to 3%, the payment frequency is quarterly and the recovery rate is $R = 40\%$. We provide in Table 1.1 model spreads and optimal parameters resulting from the four benchmark calibration processes performed on March 31st, 2008 CDX quotes. As can be seen from Table 1.1, the calibration of the three parameters on all market spreads is rather disappointing. This is not surprising, especially given the poor calibration performance of standard factor models during the crisis, when the fit is achieved on all tranches and index quotes. However, one can note that the calibration error decreases when we subsequently exclude the equity tranche quote and the index quote in Calibrations 2 and 3. Unsurprisingly, as illustrated by results from Calibration 4, the fit on equity and index spreads only is perfect. We have checked that this is actually the case for all tranches when they are jointly fitted with the index. This can be seen as a fundamental required behav-

ior of the model since we try to fit three parameters on two market quotes. Let us recall however that the base correlation framework had some difficulties to fit the super-senior tranches in the same period [see 1].

Conclusion: In this paper, we studied a particular specification of an infectious model. In our model each entity can default directly or can be infected by another defaulted entity. We analyzed the case of conditional independence of direct defaults indicators and of infections indicators. This allows us to obtain some formulas for the distribution of the number of defaults that can be applied even with large credit portfolios. This result paves the way to some operational applications regarding the pricing of CDO tranches. We then consider the fit of model parameters on CDX.NA.IG index quotes in March 2008. This allows to exploit the dynamic feature of the model and illustrate its tractability when the number of reference entities is large (and equal to 125). We can remark that, for all calibration procedures, the dependence among direct default events is exacerbated by a significant level of infectious risk, as can be expected during this distressed period.

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