Model Risk Embedded in Yield-Curve Construction Methods

Areski Cousin
ISFA, Université Lyon 1

Joint work with Ibrahima Niang

Bachelier Congress 2014

Brussels, June 5, 2014

- What is understood as a yield-curve in this presentation ?
- Term-structure construction consists in finding a function

$$T \rightarrow P(t_0, T)$$

given a small number of market quotes $S_1, \ldots, S_n$

- Market information only reliable for a small set of liquid products with standard characteristics/maturities
- We have to rely on interpolation/calibration schemes to construct the curve for missing maturities

Andersen (2007), curves based on tension splines

Figure 1: Yield Curve

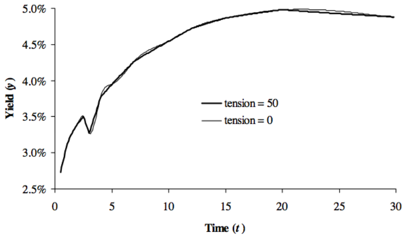
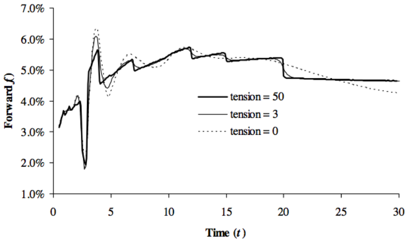
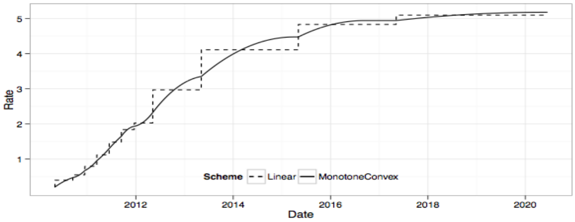


Figure 2: Forward Curve



Le Floc'h (2012), examples of one-day forward curves



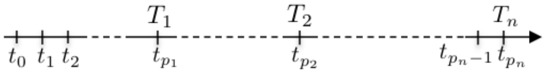
What is a good yield curve construction method ? (Hagan and West (2006))

- Ability to fit market quotes
- Arbitrage freeness
- Smoothness
- Locality of the interpolation method
- Stability of forward rate
- Consistency of hedging strategies : Locality of deltas ? Sum of sequential deltas close enough to the corresponding parallel delta ? (Le Floc'h (2012))

Market-fit condition

At time t_0, the term-structure $T \to P(t_0, T)$ is built from market quotes of standard products

- n : number of products
- $\mathbf{S} = (S_1, \ldots, S_n)$: set of market quotes at t_0
- $\mathbf{T} = (T_1, \ldots, T_n)$: corresponding set of increasing maturities



- $\mathbf{t} = (t_1, \cdots, t_m)$: payment time grid
- The two time grids $\mathbf{T}$ and $\mathbf{t}$ coincide at indices p_i such that $t_{p_i} = T_i$

Market-fit condition

Let $\mathbf{P} = (P(t_0, t_1), \ldots, P(t_0, t_m))'$ be the vector formed by the values of the curve at payment dates $t_1, \ldots, t_m$

Assumption : Linear representation of present values

Presents values of products used in the curve construction have a linear representation with respect to $\mathbf{P}$

For $i = 1, \ldots, n$

$$\sum_{k=1}^{p_i} A_{ik} P(t_0, t_k) = B_i$$

where

- $\mathbf{A} = (A_{ij})$ is a $n \times m$ matrix with positive coefficients
- $\mathbf{B} = (B_i)$ is a $n \times 1$ matrix with positive coefficients
- $\mathbf{A}$ and $\mathbf{B}$ only depend on current market quotes $\mathbf{S}$, on standard maturities $\mathbf{T}$, on payment dates $\mathbf{t}$ and on products characteristics.

Market-fit condition

Market-fit condition

The market-fit condition can be restated as a rectangular system of linear equations

$$\mathbf{A} \cdot \mathbf{P} = \mathbf{B}$$

where

- $\mathbf{P} = (P(t_0, t_1), \ldots, P(t_0, t_m))'$
- $\mathbf{A}$ is a $n \times m$ matrix with positive coefficients
- $\mathbf{B}$ is a $n \times 1$ matrix with positive coefficients
- $\mathbf{A}$ and $\mathbf{B}$ only depend on current market quotes $\mathbf{S}$, on standard maturities $\mathbf{T}$, on payment dates $\mathbf{t}$ and on products characteristics.

Market-fit condition

Example 1 : Corporate or sovereign debt yield curve

- S_i : market price (in percentage of nominal) at time t_0 of a bond with maturity T_i
- c_i : fixed coupon rate
- $t_1 < \ldots < t_{p_i} = T_i$: coupon payment dates, δ_k : year fraction of period (t_{k-1}, t_k)

$$c_i \sum_{k=1}^{p_i} \delta_k P^B(t_0, t_k) + P^B(t_0, T_i) = S_i$$

where $P^B(t_0, t_k)$ represents the price of a (fictitious default-free) ZC bond with maturity t_k

Market-fit condition

Example 2 : Discounting curve based on OIS

- S_i : par rate at time t_0 of an overnight indexed swap with maturity T_i
- $t_1 < \cdots < t_{p_i} = T_i$: fixed-leg payment dates (annual time grid)
- δ_k : year fraction of period (t_{k-1}, t_k)

$$S_i \sum_{k=1}^{p_i - 1} \delta_k P^D(t_0, t_k) + (S_i \delta_{p_i} + 1)\, P^D(t_0, T_i) = 1, \quad i = 1, ..., n$$

where $P^D(t_0, t_k)$ is the discount factor associated with maturity date t_k

Market-fit condition

Example 3 : credit curve based on CDS

- S_i : fair spread at time t_0 of a credit default swap with maturity T_i
- $t_1 < \cdots < t_p = T_i$: premium payment dates, δ_k : year fraction of period (t_{k-1}, t_k)
- R : expected recovery rate of the reference entity

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0, t_k) Q(t_0, t_k) = -(1-R) \int_{t_0}^{T_i} P^D(t_0, u) dQ(t_0, u)$$

where $u \to Q(t_0, u)$ is the $\mathcal{F}_{t_0}$-conditional (risk-neutral) survival distribution of the reference entity.

We implicitly assume here that recovery, default and interest rates are stochastically independent.

Example 3 : credit curve based on CDS (cont.)

Using an integration by parts, the survival function $u \to Q(t_0, u)$ satisfies a linear relation :

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1 - R) P^D(t_0, T_i) Q(t_0, T_i) + (1 - R) \int_{t_0}^{T_i} f^D(t_0, u) P^D(t_0, u) Q(t_0, u) du = 1 - R$$

where $f^D(t_0, u)$ is the instantaneous forward (discount) rate associated with maturity date u.

Arbitrage-free bounds

We studied two types of curves :

- Interest-rate curves : $P = P^B$ (price of zero-coupon bond), $P = P^D$ (discount factors)
- Credit curves : $P = Q$ (risk-neutral survival probability)

Arbitrage-free condition

A curve $T \to P(t_0, T)$ is said to be arbitrage-free if the two following conditions hold

- $P(t_0, t_0) = 1$
- $T \to P(t_0, T)$ is a non-increasing function

Arbitrage-free bounds

Market fit condition :

$$\sum_{k=1}^{p_1} A_{ik} P(t_0, t_k) + \cdots + \sum_{k=p_{i-1}+1}^{p_i} A_{ik} P(t_0, t_k) = B_i$$

Arbitrage-free inequalities :

$$\begin{cases} P(t_0, T_1) \leqslant P(t_0, t_k) \leqslant 1 & \text{for } 1 \leqslant k \leqslant p_1 \\ \vdots & \\ P(t_0, T_i) \leqslant P(t_0, t_k) \leqslant P(t_0, T_{i-1}) & \text{for } p_{i-1} + 1 \leqslant k \leqslant p_i - 1 \end{cases}$$

Arbitrage-free bounds

Proposition (arbitrage-free bounds)

For $i = 1, \ldots, n$,

$$P_{\min}(t_0, T_i) \leqslant P(t_0, T_i) \leqslant P_{\max}(t_0, T_i)$$

where

$$P_{\min}(t_0, T_i) = \frac{1}{A_{ip_i}} \left(B_i - \sum_{j=1}^{i-1} H_{ij} P(t_0, T_{j-1}) - (H_{ii} - A_{ip_i}) P(t_0, T_{i-1}) \right)$$

$$P_{\max}(t_0, T_i) = \frac{1}{H_{ii}} \left(B_i - \sum_{j=1}^{i-1} H_{ij} P(t_0, T_j) \right)$$

and where $H_{ij} := \sum_{k=p_{j-1}+1}^{p_j} A_{ik}$

Iterative computation of model-free bounds

- **Step 1** :

$$\widehat{P}_{\min}(t_0, T_1) \leqslant P(t_0, T_1) \leqslant \widehat{P}_{\max}(t_0, T_1)$$

where

$$\widehat{P}_{\min}(t_0, T_1) = \frac{1}{A_{1p_1}}\left(B_1 - (H_{11} - A_{1p_1})\right)$$

$$\widehat{P}_{\max}(t_0, T_1) = \frac{B_1}{H_{11}}$$

- **Step 2** : For $i = 2, \ldots, n$,

$$\widehat{P}_{\min}(t_0, T_i) \leqslant P(t_0, T_i) \leqslant \widehat{P}_{\max}(t_0, T_i)$$

where

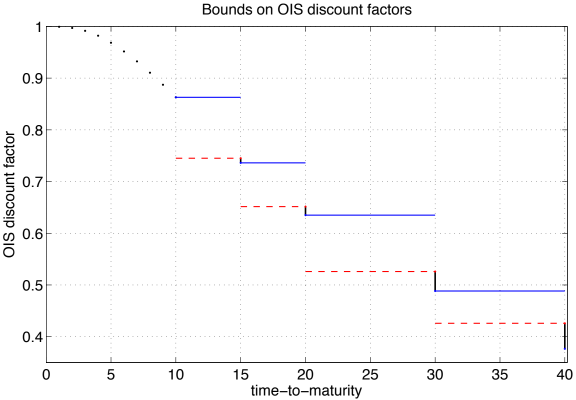
$$\widehat{P}_{\min}(t_0, T_i) = \frac{1}{A_{ip_i}}\left(B_i - \sum_{j=1}^{i-1} H_{ij}\widehat{P}_{\max}(t_0, T_{j-1}) - (H_{ii} - A_{ip_i})\widehat{P}_{\max}(t_0, T_{i-1})\right)$$

$$\widehat{P}_{\max}(t_0, T_i) = \frac{1}{H_{ii}}\left(B_i - \sum_{j=1}^{i-1} H_{ij}\widehat{P}_{\min}(t_0, T_j)\right)$$

Arbitrage-free bounds : OIS discount curves

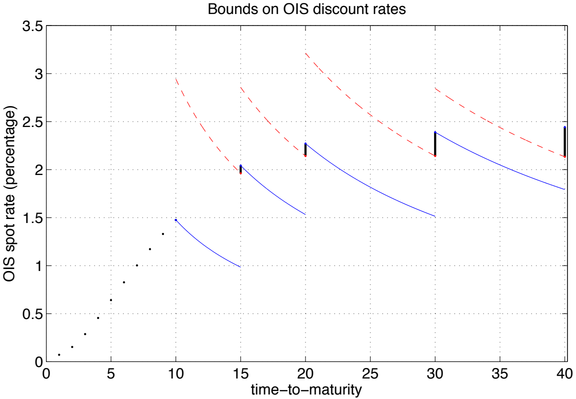
- We consider OIS par rates as of $t_0 = $ May 31st 2013
- Market quotes $\mathbf{S}$ available for $n = 14$ maturities $\mathbf{T} = (1y, 2y, \ldots, 10y, 15y, 20y, 30y, 40y)$
- $\mathbf{t} = (1y, 2y, \ldots, 10y, 11y, \ldots, 40y)$: payment time grid
- $\mathbf{A}$ is a 14×40 rectangle matrix, $\mathbf{B}$ is a 14×1 column vector
- We are looking for bounds on OIS discount factors $P^D(t_0, T_i)$, $i = 1, \ldots, n$

Bounds for OIS discount factors $P^D(t_0, T_i)$ are sharp



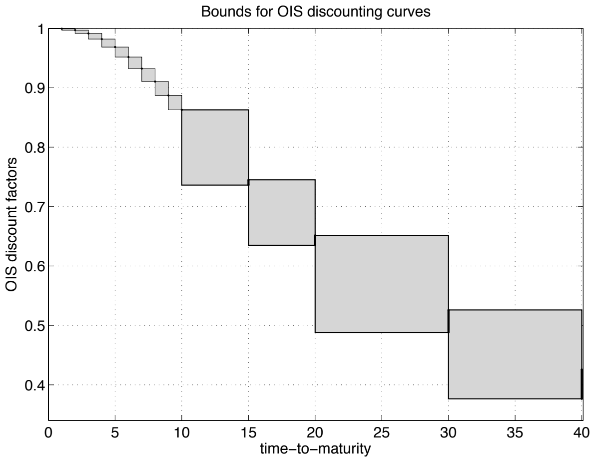
Input data : OIS swap rates as of May, 31st 2013

Bounds for the associated discount rates



Input data : OIS swap rates as of May, 31st 2013, $-\frac{1}{T}\log(P^D(t_0, T))$

Range of arbitrage-free market-consistent OIS discount curves

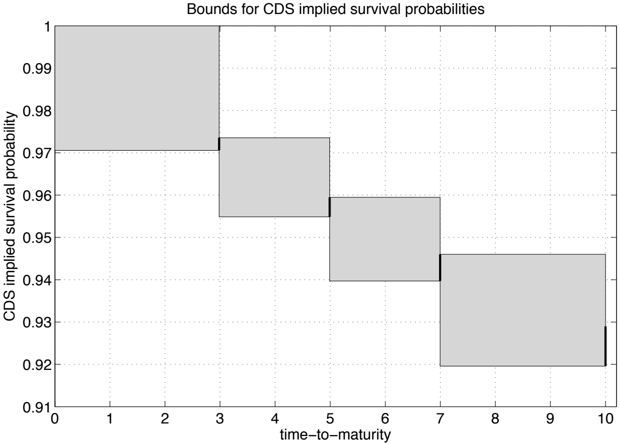


Input data : OIS swap rates as of May, 31st 2013

Arbitrage-free bounds : CDS-implied default curves

- We consider AIG CDS spreads as quoted at $t_0 = $ Dec 17, 2007
- Market quotes $\mathbf{S}$ available for $n = 4$ maturity times $\mathbf{T} = (1y, 3y, 5y, 10y)$
- $\mathbf{t} =$ the whole time interval $(0, 10y)$
- $\mathbf{A}$ is a $4 \times \infty$ rectangle matrix (the present value of CDS protection legs involves an integral instead of a sum)
- $\mathbf{B}$ is a 4×1 column vector
- We are looking for bounds on risk-neutral survival probabilities $Q(t_0, T_i)$, $i = 1, \ldots, n$

Range of market-consistent survival curves



Input data : CDS spreads of AIG as of December 17, 2007, $R = 40\%$,
$P^D(t_0, t) = \exp(-3\%(t - t_0))$

Mean-reverting term-structure models as generators of admissible yield curves

The risk-neutral dynamics of short-term interest rates (or default intensities) is assumed to follow either

- a OU process driven by a Lévy process

$$dX_t = a(b(t;\mathbf{p},\mathbf{T},\mathbf{S}) - X_t)dt + \sigma dY_{ct},$$

where Y is a Lévy process with cumulant function κ and parameter set $\mathbf{p}_L$

- or an extended CIR process

$$dX_t = a(b(t;\mathbf{p},\mathbf{T},\mathbf{S}) - X_t)dt + \sigma\sqrt{X_t}dW_t,$$

where W is a standard Browian motion

Depending on the context, $\mathbf{p} = (X_0, a, \sigma, c, \mathbf{p}_L)$ will denote the parameter set of the Lévy-OU process and $\mathbf{p} = (X_0, a, \sigma)$ the parameter set of the CIR process

How to construct admissible yield curves ?

In both cases, b is represented by a step function :

$$b(t;\mathbf{p},\mathbf{T},\mathbf{S}) = b_i(\mathbf{p},\mathbf{T},\mathbf{S}) \text{ for } T_{i-1} < t \leqslant T_i, \quad i = 1,\dots,n$$

The vector $\mathbf{b} = (b_1,\dots,b_n)$ solves a triangular system of non-linear equations.

Market-fit linear conditions

The rectangular market-fit system translates into a triangular system of non-linear equations

$$\mathbf{A}\cdot\mathbf{P}(\mathbf{b}) = \mathbf{B}$$

where

- $\mathbf{P}(\mathbf{b}) = (P(t_0,t_k;\mathbf{b}))_{k=1,\dots,m}$ is the $m\times 1$ vector of discount factors, ZC bond price or survival probabilities (depending on the context).
- $\mathbf{A}$ is a $n\times m$ matrix, $\mathbf{B}$ is a $n\times 1$ matrix
- $\mathbf{A}$ and $\mathbf{B}$ only depend on current market quotes $\mathbf{S}$, on standard maturities $\mathbf{T}$ and on products characteristics.

How to construct admissible yield curves ?

Proposition (Discount factors in the Lévy-OU approach)

Let $T_{i-1} < t \leqslant T_i$. In the Lévy-OU model, the current value of the discount factor or of an assimilated quantity with maturity time t is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$\begin{aligned} I(t_0, t, \mathbf{b}) := X_0 \phi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\xi(t - T_{k-1}) - \xi(t - T_k)\right) \\ + b_i \xi(t - T_{i-1}) + c\psi(t - t_0) \end{aligned}$$

and functions ϕ, ξ and ψ are defined by

$$\begin{aligned} \phi(s) &:= \frac{1}{a}\left(1 - e^{-as}\right) \\ \xi(s) &:= s - \phi(s) \\ \psi(s) &:= -\int_0^s \kappa\left(-\sigma\phi(s - \theta)\right) d\theta \end{aligned} \qquad (1)$$

How to construct admissible yield curves ?

Proposition (Discount factors in the CIR approach)

Let $T_{i-1} < t \leqslant T_i$. In the CIR model, the current value of the discount factor or of an assimilated quantity with maturity time t is given by

$$P(t_0, t; \mathbf{b}) := \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} X_u du\right)\right] = \exp\left(-I(t_0, t, \mathbf{b})\right)$$

where

$$I(t_0, t, \mathbf{b}) := X_0 \varphi(t - t_0) + \sum_{k=1}^{i-1} b_k \left(\eta(t - T_{k-1}) - \eta(t - T_k)\right) + b_i \eta(t - T_{i-1})$$

and functions φ and η are defined by

$$\begin{aligned} \varphi(s) &:= \frac{2(1 - e^{-hs})}{h + a + (h - a)e^{-hs}} \\ \eta(s) &:= 2a\left[\frac{s}{h + a} + \frac{1}{\sigma^2} \log \frac{h + a + (h - a)e^{-hs}}{2h}\right] \end{aligned} \quad (2)$$

where $h := \sqrt{a^2 + 2\sigma^2}$

Construction of $(b_1, \ldots, b_n)$ by a bootstrap procedure

For any $i = 1, \ldots, n$, the present value of the instrument with maturity T_i

- only depends on $b_1, \ldots, b_i$
- is a monotonic function with respect to b_i

The vector $\mathbf{b} = (b_1, \ldots, b_n)$ satisfies a triangular system of non-linear equations that can be solved recursively :

- Find b_1 as the solution of

$$\sum_{j=1}^{p_1} A_{1j} P(t_0, t_j; b_1) = B_1$$

- Assume $b_1, \ldots, b_{k-1}$ are known, find b_k as the solution of

$$\sum_{j=1}^{p_k} A_{kj} P(t_0, t_j; b_1, \ldots, b_k) = B_k$$

Proposition (smoothness condition)

A curve $t \to P(t_0, t)$ constructed from the previous approach is of class $\mathcal{C}^1$ and the corresponding forward curve (or default density function) is continuous.

Proof : Let $b(\cdot)$ be a deterministic function of time, instantaneous forward rates are such that

- Lévy-driven OU

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \int_{t_0}^{t} e^{-a(t-u)} b(u) du - c\kappa(-\sigma\phi(t - t_0))$$

  where ϕ is defined by (1)

- extended CIR

$$f^{CIR}(t_0, t) = X_0 \varphi'(t - t_0) + a \int_{t_0}^{t} \varphi'(t - u) b(u) du$$

  where φ' is the derivative of φ given by (2)

How to construct admissible yield curves ?

Assume that a curve has been constructed from a Lévy-OU term-structure model with positive parameters $(X_0, a, \sigma, c, \mathbf{p}_L)$:

$$f(t_0, t) = X_0 e^{-a(t-t_0)} + a \sum_{k=1}^{i-1} b_k \left(\phi(t - T_{k-1}) - \phi(t - T_k)\right) + ab_i\phi(t - T_{i-1}) - c\kappa(-\sigma\phi(t - t_0))$$

for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

Proposition (arbitrage-free condition in the Lévy-OU approach)

Assume that the derivative of the Lévy cumulant κ' exists and is strictly monotonic on $(-\infty, 0)$. The curve is arbitrage-free on the time interval (t_0, T_n) if and only if, for any $i = 1, \ldots, n$, $f(t_0, T_i) > 0$ and one of the following condition holds :

- $\frac{\partial f}{\partial t}(t_0, T_{i-1})\frac{\partial f}{\partial t}(t_0, T_i) \geq 0$
- $\frac{\partial f}{\partial t}(t_0, T_{i-1})\frac{\partial f}{\partial t}(t_0, T_i) < 0$ and $f(t_0, t_i) > 0$ where t_i is such that $\frac{\partial f}{\partial t}(t_0, t_i) = 0$,

How to construct admissible yield curves ?

Assume that a curve has been constructed from an extended CIR term-structure model with positive parameters (X_0, a, σ) :

$$f^{CIR}(t_0, t) = X_0 \varphi'(t - t_0) + a \sum_{k=1}^{i-1} b_k \left(\varphi(t - T_{k-1}) - \varphi(t - T_k)\right) + a b_i \varphi(t - T_{i-1})$$

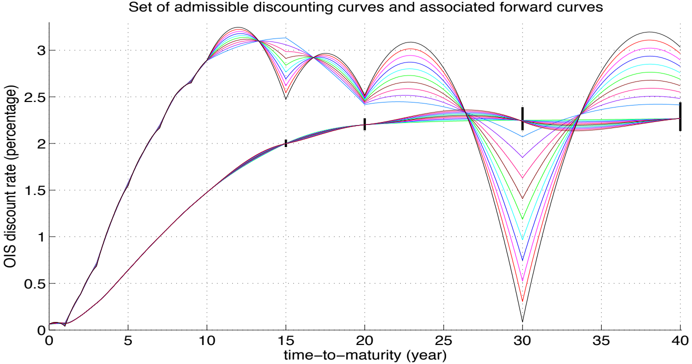
for any $T_{i-1} \leq t \leq T_i$, $i = 1, \ldots, n$.

Proposition (arbitrage-free condition in the CIR approach)

The constructed curve is arbitrage-free if, for any $i = 1, \cdots, n$, the implied b_i is positive

Set of admissible OIS discount and forward curves : Lévy-OU short rates

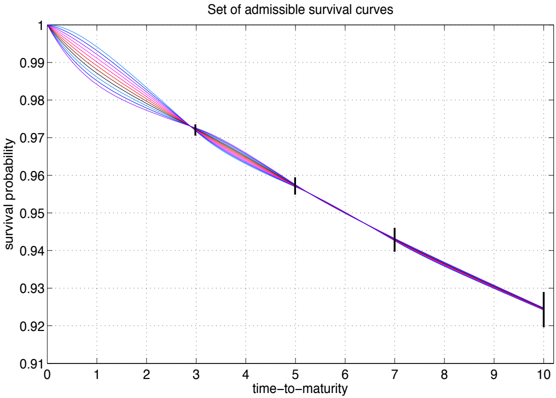
Parameters : $a = 0.01$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M). The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/50$bps (mean jump size of 50 bps). $c = \{1, 10, 20, \ldots, 100\}$



Input data : OIS swap rates as of May, 31st 2013

Set of admissible survival curves : CIR intensities

Parameters : $a = \sigma = 1$, $100 \cdot X_0 = \{0.01, 0.25, 0.49, 0.73, 0.97, 1.21, 1.45, 1.69, 1.94, 2.18, 2.42\}$



Input data : CDS spreads of AIG as of December 17, 2007, $R = 40\%$,
$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$

Perspectives

The proposed framework could be extended or used in several directions :

- Yield-curve diversity impact on present values (PV) and hedging stategies ?

  $$\max_{i,j} \|PV(C_i) - PV(C_j)\|_p$$

  where the max is taken over all couples of admissible curves (C_i, C_j)

- Risk management in the presence of uncertain parameters ?

  $$dX_t = \tilde{a}(b(t; \tilde{a}, \tilde{\sigma}, \mathbf{T}, \mathbf{S}) - X_t)dt + \tilde{\sigma}\sqrt{X_t}dW_t,$$

  where $\text{Range}(\tilde{a}, \tilde{\sigma}) \subset \{(a, \sigma) \mid b(t; a, \sigma, \mathbf{T}, \mathbf{S}) \geq 0 \; \forall t\}$

- Extension to multicurve environments ?

- Impact on the assessment of counterparty credit risk (CVA, EE, EPE, ...) ?

References (curve construction methods)

- Andersen, 2007, *Discount curve construction with tension splines*
- Ametrano and Bianchetti, 2009, *Bootstrapping the illiquidity*
- Chibane, Selvaraj and Sheldon, 2009, *Building curves on a good basis*
- Fries, 2013, *Curves and term structure models. Definition, calibration and application of rate curves and term structure market models*
- Hagan and West, 2006, *Interpolation methods for curve construction*
- Iwashita, 2013, *Piecewise polynomial interpolations*
- Jerassy-Etzion, 2010, *Stripping the yield curve with maximally smooth forward curves*
- Kenyon and Stamm, 2012, *Discounting, LIBOR, CVA and funding : Interest rate and credit pricing*
- Le Floc'h, 2012, *Stable interpolation for the yield curve*

References (model risk)

- Branger and Schlag, 2004, *Model risk : A conceptual framework for risk measurement and hedging*
- Cont, 2006, *Model uncertainty and its impact on the pricing of derivative instruments*
- Davis and Hobson, 2004, *The range of traded option prices*
- Derman, 1996, *Model risk*
- Eberlein and Jacod, 1997, *On the range of option prices*
- El Karaoui, Jeanblanc and Shreve, 1998, *Robustness of the Black and Scholes formula*
- Green and Figlewski, 1999, *Market risk and model risk for a financial institution writing options*
- Hénaff, 2010, *A normalized measure of model risk*
- Morini, 2010, *Understanding and managing model risk*

Cumulant function of some Lévy processes

	Cumulant
Brownian motion	$\kappa(\theta) = \frac{\theta^2}{2}$
Gamma process	$\kappa(\theta) = -\log\left(1 - \frac{\theta}{\lambda}\right)$
Inverse Gaussian process	$\kappa(\theta) = \lambda - \sqrt{\lambda^2 - 2\theta}$