On the range of admissible term-structures

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Introduction

What is understood as a term-structure in this presentation?

What is it used for?
Term-structures are constructed from market quotes of fixed income, fx or default-risky securities.

Information provided by the market is reliable only for a small set of liquid products with standard characteristics/maturities.

We have to rely on interpolation/calibration schemes to construct the curve for missing maturities.

A variety of curve construction methods exists: no consensus towards a particular best practice in all circumstances.
Andersen (2007), curves based on tension splines

Le Floc’h (2012), examples of one-day forward curves
What can be defined as a good yield curve construction method? (Hagan and West (2006))

- Ability to fit market quotes
- Arbitrage freeness
- Smoothness
- Locality of the interpolation method
- Stability of forward rate
- Consistency of hedging strategies: Locality of deltas? Sum of sequential deltas close enough to the corresponding parallel delta? (Le Floc’h (2012))
Interestingly, there is a pretty large recent literature on the subject of yield-curve construction methods


And a flourishing literature on model risk

Introduction

Arbitrage-free curve
A curve is said to be arbitrage-free if

- **IR curves**: the forward rates are non-negative or equivalently, the (pseudo) discount factors are nonincreasing with respect to time-to-maturities
- **Credit**: the curve is associated with a well-defined default distribution function

Smoothness condition
A curve is said to be smooth if

- **IR curves**: the instantaneous forward rates exist for all maturities and are continuous.
- **Credit**: the default density function exists and is continuous.
A yield curve is said to be admissible if it satisfies the following constraints:

- The input data set is perfectly reproduced by the curve
- The curve is arbitrage-free
- The curve satisfies the smoothness condition
Introduction

We then address the following questions:

- Is it possible to estimate the size of admissible curves? and how?
- How does the range/diversity of admissible curves affect the present value of products with non-standard characteristics?

We develop a framework in which it is possible to measure the diversity of yield curves with some specific features.
Instruments used for curve construction

Assumption 1: pseudo-linear representation of present values

Products used in the curve construction have presents values that can be expressed as linear combination of some elementary quantities such as zero-coupon prices, discount factors, Ibor forward rates or survival probabilities.

Example 1: Corporate or sovereign debt yield curve

- $S$: market price (in percentage of nominal) at time $t_0$ of a bond with maturity $T$
- $c$: fixed coupon rate
- $t_1 < \ldots < t_p = T$: coupon payment dates, $\delta_k$: year fraction corresponding to period $(t_{k-1}, t_k)$

$$c \sum_{k=1}^{p} \delta_k P^B(t_0, t_k) + P^B(t_0, T) = S$$

where $P^B(t_0, t_k)$ represents the price of a (fictitious default-free issuer-dependent) ZC bond with maturity $t_k$
Example 2: Discounting curve based on OIS

- $S^{OIS}$: par rate at time $t_0$ of an overnight indexed swap with maturity $T$
- $t_1 < \cdots < t_p = T$: fixed-leg payment dates
- $\delta_k$: year fraction corresponding to period $(t_{k-1}, t_k)$

$$S^{OIS} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = 1 - P^D(t_0, T)$$

where $P^D(t_0, t_k)$ is the discount factor associated with maturity date $t_k$
Example 3 : Forward curve based on fixed-vs-Ibor-floating IRS

- \( S_{\text{IRS}} \): par rate at time \( t_0 \) of an interest rate swap with maturity \( T \) and tenor \( j \) (typically, \( j = 3\text{M} \) or \( j = 6\text{M} \))
- \( t_1 < \cdots < t_p = T \): fixed-leg payment dates, \( \delta_k \): year fraction corresponding to period \((t_{k-1}, t_k)\)
- \( t = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_q = T \): floating-leg payment dates, \( \tilde{\delta}_i \): year fraction of \((\tilde{t}_{i-1}, \tilde{t}_i)\)

\[
S_{\text{IRS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) = \sum_{i=1}^{q} P^D(t_0, \tilde{t}_i)\tilde{\delta}_i F_j(t_0, \tilde{t}_i)
\]

where \( F_j(t_0, \tilde{t}_i) \) is the forward Libor or Euribor rate defined as the fixed rate to be exchanged at time \( \tilde{t}_i \) against the \( j \)-tenor Libor or Euribor rate established at time \( \tilde{t}_{i-1} \) so that the swap has zero value at time \( t_0 \)
Example 4: credit curve based on CDS

- $S^{\text{CDS}}$: fair spread at time $t_0$ of a credit default swap with maturity $T$
- $t_1 < \cdots < t_p = T$: premium payment dates, $\delta_k$: year fraction corresponding to period $(t_{k-1}, t_k)$
- $R$: expected recovery rate of the reference entity

$$S^{\text{CDS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) Q(t_0, t_k) = - (1 - R) \int_{t_0}^{T} P^D(t_0, u) dQ(t_0, u)$$

where $u \rightarrow Q(t_0, u)$ is the $\mathcal{F}_{t_0}$-conditional (risk-neutral) survival distribution of the reference entity.

We implicitly assume here that recovery, default and interest rates are stochastically independent.
Example 4 : credit curve based on CDS (cont)

Using an integration by parts, the survival function $u \rightarrow Q(t_0, u)$ satisfies a linear relation:

$$S^{\text{CDS}} \sum_{k=1}^{p} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1 - R) P^D(t_0, T) Q(t_0, T)$$

$$+ (1 - R) \int_{t_0}^{T} f^D(t_0, u) P^D(t_0, u) Q(t_0, u) du = 1 - R$$

where $f^D(t_0, u)$ is the instantaneous forward (discount) rate associated with maturity date $u$. 
Proposition (admissible curves form a convex set)

Under Assumption I, the set of admissible yield-curve is convex.

This derives immediately from the definition of admissible curves and the linear representation of present values.

Proposition

Under Assumption I, the set of admissible yield-curves is characterized by the convex hull of the extreme points of its closure.

Identifying the set of admissible yield-curves amounts to identify its convex hull.
The proof follows from successive applications of Ascoli-Arzelà theorem and Krein-Milman theorem.

### Ascoli-Arzelà theorem

Let $(X, d)$ be a compact space. A subset $F$ of $C(X)$ is relatively compact if and only if $F$ is equibounded and equicontinuous.

We have to prove that $F$ is equibounded and equicontinuous.

### Krein-Milman theorem

Let $X$ be a locally convex topological vector space (assumed to be Hausdorff or separable), and let $K$ be a compact convex subset of $X$. Then $K$ is the closed convex hull of its extreme points.
Arbitrage-free bounds for OIS discount curves

- We observe OIS par rates $S_1, \cdots, S_n$ for maturities $T_1 < \cdots < T_n$.

- Let $t = t_0 < t_1 < \cdots < t_{p_n} = T_n$ be the annual time grid up to time $T_n$.

- The set of indices $(p_i)$ is such that $t_{p_i} = T_i$ for $i = 1, \ldots, n$.

$$S_i \sum_{k=1}^{p_i-1} \delta_k P^D(t_0, t_k) + (S_i \delta_{p_i} + 1) P^D(t_0, T_i) = 1, \quad i = 1, \ldots, n$$

- Let $i_0$ be the smallest index such that $T_i_0 \neq t_{i_0}$ ($i_0 = 11$ in our applications)

- Define $H_i := \sum_{k=p_i-1+1}^{p_i-1} \delta_k$, for $i = i_0, \ldots, n$
Proposition (arbitrage-free bounds for discount factors)

\[ P^D(t_0, T_1) = \frac{1}{1 + S_1 \delta_1}, \]

\[ P^D(t_0, T_i) = \frac{1}{1 + S_i \delta_i} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right), \quad i = 2, \ldots, i_0 - 1 \]

For \( i = i_0, \ldots, n, \)

\[ P^D_{\text{min}}(t_0, T_i) \leq P^D(t_0, T_i) \leq P^D_{\text{max}}(t_0, T_i) \]

where

\[ P^D_{\text{min}}(t_0, T_i) = \frac{1}{1 + S_i \delta_{p_i}} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - (1 - S_{i-1} H_i) P^D(t_0, T_{i-1}) \right) \right) \]

\[ P^D_{\text{max}}(t_0, T_i) = \frac{1}{1 + S_i (H_i + \delta_{p_i})} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right) \]
Proof:

For any $i = i_0, \ldots, n$, the previous rectangular system of OIS present values can be simplified:

$$
\frac{S_i}{S_{i-1}} \left(1 - P^D(t_0, T_{i-1})\right) + S_i \sum_{k=p_{i-1}+1}^{p_i-1} \delta_k P^D(t_0, t_k) + (1 + S_i \delta_p_i) P^D(t_0, T_i) = 1
$$

The bounds derive from the following system of arbitrage-free inequalities:

$$
\begin{cases}
P^D(t_0, T_{i_0}) \leq P^D(t_0, t_k) \leq P^D(t_0, T_{i_0-1}) & \text{for } p_{i_0-1} + 1 \leq k \leq p_{i_0} - 1 \\
\vdots & \\
P^D(t_0, T_i) \leq P^D(t_0, t_k) \leq P^D(t_0, T_{i-1}) & \text{for } p_{i-1} + 1 \leq k \leq p_i - 1
\end{cases}
$$

These bounds cannot be computed since we do not know the discount factors $P^D(t_0, T_i)$ for $i = i_0, \ldots, n$. 
Iterative computation of model-free bounds

- **Step 1**: For \( i = 1, \ldots, i_0 - 1, \)

  \[
  P^D(t_0, T_i) = \frac{1}{1 + S_i \delta_i} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P^D(t_0, T_{i-1}) \right) \right)
  \]

- **Step 2**: For \( i = i_0, \ldots, n, \)

  \[
  P_{\text{min}}(T_i) \leq P^D(t_0, T_i) \leq P_{\text{max}}(T_i)
  \]

  where

  \[
  P_{\text{min}}(T_i) = \frac{1}{1 + S_i \delta_{p_i}} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - (1 - S_{i-1} H_i) P_{\text{min}}(T_{i-1}) \right) \right)
  \]

  \[
  P_{\text{max}}(T_i) = \frac{1}{1 + S_i (H_i + \delta_{p_i})} \left( 1 - \frac{S_i}{S_{i-1}} \left( 1 - P_{\text{max}}(T_{i-1}) \right) \right)
  \]
The previous model-free bounds are sharp

**Input data** : OIS swap rates as of May, 31st 2013
Arbitrage-free bounds for OIS discount curves

Corresponding model-free bounds on discount rates

Input data: OIS swap rates as of May, 31st 2013
Range of arbitrage-free market-consistent OIS discount curves

Input data: OIS swap rates as of May, 31st 2013
Proposition (detecting arbitrage opportunities)

An arbitrage opportunity can be detected in the data set \((S_i)_{i=1,...,n}\) at the first index \(i\) such that

\[
S_i < \left( \frac{1}{S_{i-1}} + \delta_i \frac{P^D(t_0, T_{i-1})}{1 - P^D(t_0, T_{i-1})} \right)^{-1}, \quad i = 2, \ldots, i_0 - 1,
\]

\[
S_i < \left( \frac{1}{S_{i-1}} + (H_i + \delta_{p_i}) \frac{P_{\text{max}}(T_{i-1})}{1 - P_{\text{max}}(T_{i-1})} \right)^{-1}, \quad i = i_0, \ldots, n.
\]

Proof:

For \(i = 2, \ldots, i_0 - 1\), the inequality on \(S_i\) leads to \(P^D(t_0, T_i) > P^D(t_0, T_{i-1})\)

For \(i = i_0, \ldots, n\), the inequality on \(S_i\) leads to \(P^D_{\text{min}}(t_0, T_i) > P^D_{\text{max}}(t_0, T_i)\)
Corollary (increasing OIS par rates are arbitrage-free)

An increasing sequence of OIS par rates $S_1 \leq \cdots \leq S_n$ is arbitrage-free: there always exits an arbitrage-free discount curve which is compatible with this sequence.
We observe CDS fair spreads \( S_1, \ldots, S_n \) for maturities \( T_1 < \cdots < T_n \).

Let \( t = t_0 < t_1 < \cdots < t_{pn} = T_n \) be the time grid corresponding to premium payment dates.

The set of indices \((p_i)\) is such that \( p_0 = 1 \) and \( t_{p_i} = T_i \) for \( i = 1, \ldots, n \).

For \( i = 1, \ldots, n \),

\[
S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0, t_k) Q(t_0, t_k) + (1 - R) P^D(t_0, T) Q(t_0, T) \\
+ (1 - R) \int_{t_0}^{T_i} f^D(t_0, t) P^D(t_0, t) Q(t_0, t) dt = 1 - R
\]
Proposition (arbitrage-free bounds for survival probabilities)

For \( i = 1, \ldots, n \),

\[
Q_{\text{min}}(t_0, T_i) \leq Q(t_0, T_i) \leq Q_{\text{max}}(t_0, T_i)
\]

where

\[
Q_{\text{min}}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i} ((1 - R)M_k + S_iN_k) Q(t_0, T_{k-1})}{P^D(t_0, T_i)(1 - R + S_i\delta_{p_i})},
\]

\[
Q_{\text{max}}(t_0, T_i) = \frac{1 - R - \sum_{k=1}^{i-1} ((1 - R)M_k + S_iN_k) Q(t_0, T_k)}{P^D(t_0, T_{i-1})(1 - R) + S_i(N_i + \delta_{p_i}P^D(t_0, T_i))},
\]

with \( M_i := P^D(t_0, T_{i-1}) - P^D(t_0, T_i) \) and \( N_i := \sum_{k=p_{i-1}}^{p_i-1} \delta_k P^D(t_0, t_k) \).
Proof:

For any $i = 1, \ldots, n$, the proof is based on CDS present value representations as linear combinations of survival probabilities and application of the following system of “arbitrage-free inequalities”:

\[
\begin{align*}
Q(t_0, T_1) \leq Q(t_0, t) \leq 1 & \quad \text{for } t_0 \leq t < T_1, \\
& \vdots \\
Q(t_0, T_i) \leq Q(t_0, t) \leq Q(t_0, T_{i-1}) & \quad \text{for } T_{i-1} \leq t < T_i
\end{align*}
\]

These bounds cannot be computed explicitly since we do not know the survival probabilities $Q(t, T_i)$ with certainty for $i = 1, \ldots, n$. 

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On the range of admissible term-structures
Iterative computation of model-free bounds

- For \( i = 1, \ldots, n \), compute recursively

\[
Q_{\text{min}}(T_i) \leq Q(t_0, T_i) \leq Q_{\text{max}}(T_i)
\]

where

\[
Q_{\text{min}}(T_i) = \frac{1 - R - \sum_{k=1}^{i} ((1 - R) M_k + S_i N_k) Q_{\text{max}}(T_{k-1})}{P^D(t, T_i)(1 - R + S_i \delta p_i)}
\]

\[
Q_{\text{max}}(T_i) = \frac{1 - R - \sum_{k=1}^{i-1} ((1 - R) M_k + S_i N_k) Q_{\text{min}}(T_k)}{P^D(t, T_{i-1})(1 - R) + S_i (N_i + \delta p_i P^D(t, T_i))}
\]
Input data: CDS spreads of AIG as of December 17, 2007, $R = 40\%$, 

$$P^D(t_0, t) = \exp(-3\%(t - t_0))$$
Arbitrage-free bounds for survival curves

Bounds sensitivity with respect to the recovery rate assumption

Input data: CDS spreads of AIG as of December 17, 2007,
\[ P^D(t_0, t) = \exp(-3\% (t - t_0)) \]
The yield-curve is built from market quotes of a set of standard products:

- $t_0$: quotation date
- $T = (T_1, \ldots, T_n)$: set of increasing standard maturities, $T_0 = t_0$
- $S = (S_1, \ldots, S_n)$: corresponding set of market quotes at $t_0$

We assume that present values can be expressed as linear combination of generic elementary quantities comparable to discount factors:

- $P = P^B$, zero-coupon prices as in Example 1
- $P = P^D$, discount factors as in Example 2
- $P = Q$, risk-neutral survival probabilities as in Example 4

In this presentation, we do not treat the case of elementary quantities comparable to forward rates as in Example 3.
Mean-reverting term-structure models as generators of admissible yield curves

The risk-neutral dynamics of (default-free) interest rates or of default intensities is assumed to follow either

- a OU process driven by a Lévy process
  \[ dX_t = a(b(t; p, T, S) - X_t)dt + \sigma dY_{ct}, \]
  where \( Y \) is a Lévy process with cumulant function \( \kappa \) and parameter set \( p_L \)

- or an extended CIR process
  \[ dX_t = a(b(t; p, T, S) - X_t)dt + \sigma \sqrt{X_t} dW_t, \]
  where \( W \) is a standard Brownian motion

Depending on the context, \( p = (X_0, a, \sigma, c, p_L) \) will denote the parameter set of the Lévy-OU process and \( p = (X_0, a, \sigma) \) the parameter set of the CIR process
How to construct admissible yield curves?

In both cases, \( b \) is represented by a step function:

\[
b(t; p, T, S) = b_i(p, T, S) \quad \text{for} \quad T_{i-1} < t \leq T_i, \quad i = 1, \ldots, n
\]

The vector \( b = (b_1, \ldots, b_n) \) solves the following pseudo-linear system.

**Market-fit linear conditions**

The market-fit condition can be restated as a pseudo-linear system

\[
A \cdot P(b) = B
\]

where

- \( P(b) = (P(t_0, t_k; b))_{k=1, \ldots, m} \) is the \( m \times 1 \) vector of elementary quantities that appear in the present value formula of instruments used to build the curve (see Examples 1 to 4).
- \( A \) is a \( n \times m \) matrix, \( B \) is a \( n \times 1 \) matrix
- \( A \) and \( B \) only depend on current market quotes \( S \), on standard maturities \( T \) and on products characteristics.
How to construct admissible yield curves?

Proposition (Discount factors in the Lévy-OU approach)

Let \( T_{i-1} < t \leq T_i \). In the Lévy-OU model, the current value of the discount factor or of an assimilated quantity with maturity time \( t \) is given by

\[
P(t_0, t; b) := \mathbb{E} \left[ \exp \left( - \int_{t_0}^{t} X_u du \right) \right] = \exp(-I(t_0, t, b))
\]

where

\[
I(t_0, t, b) := X_0 \phi(t-t_0) + \sum_{k=1}^{i-1} b_k \left( \xi(t-T_{k-1}) - \xi(t-T_k) \right)
\]

\[
+ b_i \xi(t-T_{i-1}) + c \psi(t-t_0)
\]

and functions \( \phi, \xi \) and \( \psi \) are defined by

\[
\phi(s) := \frac{1}{a} \left( 1 - e^{-as} \right)
\]

\[
\xi(s) := s - \phi(s)
\]

\[
\psi(s) := - \int_0^s \kappa(-\sigma \phi(s-\theta)) d\theta
\]
Proposition (Discount factors in the CIR approach)

Let $T_{i-1} < t \leq T_i$. In the CIR model, the current value of the discount factor or of an assimilated quantity with maturity time $t$ is given by

$$P(t_0, t; b) := \mathbb{E} \left[ \exp \left( - \int_{t_0}^{t} X_u du \right) \right] = \exp (-I(t_0, t, b))$$

where

$$I(t_0, t, b) := X_0 \varphi(t - t_0) + \sum_{k=1}^{i-1} b_k \left( \eta(t - T_{k-1}) - \eta(t - T_k) \right) + b_i \eta(t - T_{i-1})$$

and functions $\varphi$ and $\eta$ are defined by

$$\varphi(s) := \frac{2(1 - e^{-hs})}{h + a + (h - a)e^{-hs}}$$

$$\eta(s) := 2a \left[ \frac{s}{h + a} + \frac{1}{\sigma^2} \log \frac{h + a + (h - a)e^{-hs}}{2h} \right]$$

where $h := \sqrt{a^2 + 2\sigma^2}$.
How to construct admissible yield curves?

Construction of \((b_1, \ldots, b_n)\) by a bootstrap procedure

For any \(i = 1, \ldots, n\), the present value of the instrument with maturity \(T_i\)
- only depends on \(b_1, \ldots, b_i\)
- is a monotonic function with respect to \(b_i\)

The vector \(b = (b_1, \ldots, b_n)\) satisfies a triangular system of non-linear equations that can be solved recursively:
- Find \(b_1\) as the solution of
  \[
  \sum_{j=1}^{p_1} A_{1j} P(t_0, t_j; b_1) = B_1
  \]
- Assume \(b_1, \ldots, b_{k-1}\) are known, find \(b_k\) as the solution of
  \[
  \sum_{j=1}^{p_k} A_{kj} P(t_0, t_j; b_1, \ldots, b_k) = B_k
  \]
How to construct admissible yield curves?

Proposition (smoothness condition)

A curve \( t \to P(t_0, t) \) constructed from the previous approach satisfies the smoothness condition: it is of class \( C^1 \) and the corresponding forward curve (or default density function) is continuous.

Proof: Let \( b(\cdot) \) be a deterministic function of time, instantaneous forward rates are such that

- Lévy-driven OU

\[
f(t_0, t) = X_0 e^{-a(t-t_0)} + a \int_{t_0}^{t} e^{-a(t-u)} b(u) du - cK(-\sigma \phi(t - t_0))
\]

where \( \phi \) is defined by (1)

- extended CIR

\[
f^{CIR}(t_0, t) = X_0 \phi'(t - t_0) + a \int_{t_0}^{t} \phi'(t - u) b(u) du
\]

where \( \phi' \) is the derivative of \( \phi \) given by (2)
How to construct admissible yield curves?

Assume that a curve has been constructed from a \textit{Lévy-OU term-structure model} with positive parameters \((X_0, a, \sigma, c, \mathbf{p}_L)\):

\[
f(t_0, t) = X_0 e^{-a(t-t_0)} + a \sum_{k=1}^{i-1} b_k \left( \phi(t - T_{k-1}) - \phi(t - T_k) \right) + ab_i \phi(t - T_{i-1}) - c\kappa(-\sigma \phi(t - t_0))
\]

for any \(T_{i-1} \leq t \leq T_i, \ i = 1, \ldots, n.\)

**Proposition (arbitrage-free condition in the Lévy-OU approach)**

Assume that the derivative of the Lévy cumulant \(\kappa'\) exists and is strictly monotonic on \((-\infty, 0)\). The curve is arbitrage-free on the time interval \((t_0, T_n)\) if and only if, for any \(i = 1, \ldots, n\), \(f(t_0, T_i) > 0\) and one of the following condition holds:

- \(\frac{\partial f}{\partial t}(t_0, T_{i-1}) \frac{\partial f}{\partial t}(t_0, T_i) \geq 0\)
- \(\frac{\partial f}{\partial t}(t_0, T_{i-1}) \frac{\partial f}{\partial t}(t_0, T_i) < 0\) and \(f(t_0, t_i) > 0\) where \(t_i\) is such that \(\frac{\partial f}{\partial t}(t_0, t_i) = 0\).
How to construct admissible yield curves?

Assume that a curve has been constructed from an extended CIR term-structure model with positive parameters \((X_0, a, \sigma)\):

\[
f^{\text{CIR}}(t_0, t) = X_0 \varphi'(t - t_0) + a \sum_{k=1}^{i-1} b_k (\varphi(t - T_{k-1}) - \varphi(t - T_k)) + ab_i \varphi(t - T_{i-1})
\]

for any \(T_{i-1} \leq t \leq T_i, \ i = 1, \ldots, n\).

**Proposition (arbitrage-free condition in the Lévy-OU approach)**

The constructed curve is arbitrage-free if, for any \(i = 1, \ldots, n\), the implied \(b_i\) is positive.
How to construct admissible yield curves?

Set of admissible OIS discount and forward curves: Lévy-OU short rates

Parameters: $a = 0.01$, $\sigma = 1$, $X_0 = 0.063\%$ (fair rate of IRS vs OIS 1M). The Lévy driver is a Gamma subordinator with parameter $\lambda = 1/50\text{bps}$ (mean jump size of 50 bps). $c = \{1, 10, 20, \ldots, 100\}$

Input data: OIS swap rates as of May, 31st 2013
How to construct admissible yield curves?

Arbitrage-free bounds used to generate a wider range of admissible curves

Parameters: CIR short rates with \( a = 5 \), \( \sigma = 1 \), \( X_0 = 0.063\% \) (fair rate of IRS vs OIS 1M).

Input data: OIS swap rates as of May, 31st 2013
How to construct admissible yield curves?

Set of admissible survival curves: CIR intensities

Parameters: \( a = \sigma = 1, \ 100X_0 = \{0.01, 0.25, 0.49, 0.73, 0.97, 1.21, 1.45, 1.69, 1.94, 2.18, 2.42\} \)

Input data: CDS spreads of AIG as of December 17, 2007, \( R = 40\% \),

\[ P^D(t_0, t) = \exp(-3\%(t - t_0)) \]
How to construct admissible yield curves?

Set of admissible survival curves: Lévy-OU intensities

Parameters: \( a \) is Uniform on \([0.5, 10]\), \( c \) is Uniform on \([1, 50]\) \((c:\text{mean number of jumps per year})\), \( \sigma = 1 \). The Lévy driver is a Gamma subordinator with parameter \( \lambda = 1/2 \text{bps} \) \(\text{(mean jump size of 2 bps)}\), \( X_0 \) is bootstrapped with \( b_1 \) in such a way that \( X_0 = b_1 \).

Input data: CDS spreads of AIG as of December 17, 2007
The proposed framework could be extended or used in several directions:

- Yield-curve diversity impact on present values (PV) and hedging strategies?
  \[
  \max_{i,j} \| PV(C_i) - PV(C_j) \|_p
  \]
  where the max is taken over all couples of admissible curves \((C_i, C_j)\)

- Sensitivity analysis in the presence of uncertain parameters?
  \[
  dX_t = \tilde{a}(b(t; \tilde{a}, \tilde{\sigma}, T, S) - X_t)dt + \tilde{\sigma}\sqrt{X_t}dW_t,
  \]
  where \(\text{Range}(\tilde{a}, \tilde{\sigma}) \subset \{(a, \sigma) \mid b(t; a, \sigma, T, S) \geq 0 \ \forall t\}\)

- Extension to a multicurve environment?

- Impact on the assessment of counterparty credit risk (CVA, EE, EPE, ...)?
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Cumulant function of some Lévy processes

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<tr>
<th>Process</th>
<th>Cumulant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brownian motion</td>
<td>$\kappa(\theta) = \frac{\theta^2}{2}$</td>
</tr>
<tr>
<td>Gamma process</td>
<td>$\kappa(\theta) = -\log \left(1 - \frac{\theta}{\lambda}\right)$</td>
</tr>
<tr>
<td>Inverse Gaussian process</td>
<td>$\kappa(\theta) = \lambda - \sqrt{\lambda^2 - 2\theta}$</td>
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