On Multivariate Extensions of Value-at-Risk and Conditional-Tail-Expectation

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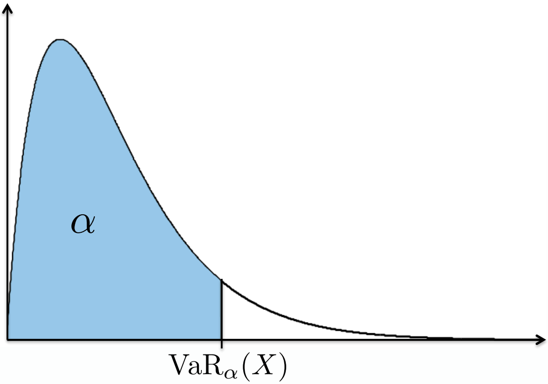
- A. Cousin, E. Di Bernardino, *On Multivariate Extensions of Value-at-Risk*, *Journal of Multivariate Analysis*

- A. Cousin, E. Di Bernardino, *On Multivariate Extensions of Conditional-Tail-Expectation*, in preparation

Value-at-Risk measure



Given an univariate continuous and strictly monotonic loss distribution function F_X,

$$\mathrm{VaR}_\alpha(X) = Q_X(\alpha) = F_X^{-1}(\alpha), \quad \forall\, \alpha \in (0,1).$$

Multivariate risk problems require multivariate measures

- Financial risks are strongly interconnected and cannot be managed individually
- Construction of risk measures that account both for marginal effects and dependence between risks
- Multivariate risk measures involve in different applications
  1) Capital allocation problem
  2) Measures of systemic risk
  3) Measures for risks with heterogeneous characteristics

Capital allocation problem

- $X = (X_1, \ldots, X_d)$: risk exposures of a given financial institution
- X_j : risk exposure of underlying entity j (could be a subsidiary, an operational branch, a risk category)
- Capital charge is measured from the aggregated risk

$$L = X_1 + \cdots + X_d$$

- What is the contribution of each subsidiary ?

Euler (or Shapley-Aumann) allocation rule involves both X_i and L:

$$\mathrm{VaR}_\alpha(X_i \mid L) = \mathbb{E}[\, X_i \,|\, L = \mathrm{VaR}_\alpha(L)\,],\ i = 1, \ldots, d$$

$$\mathrm{ES}_\alpha(X_i \mid L) = \mathbb{E}[\, X_i \,|\, L \geq \mathrm{VaR}_\alpha(L)\,],\ i = 1, \ldots, d$$

Scaillet (2004), Tache (2008)

Other types of multivariate risk measures

Other risk measures based on function min or max have been proposed:

- $\mathrm{CTE}_{\alpha}^{\min}(X_i) = \mathbb{E}[\,X_i \,|\, X_{(1)} \geq Q_{X_{(1)}}(\alpha)\,]$ where $X_{(1)} = \min\{X_1, \ldots, X_d\}$
- $\mathrm{CTE}_{\alpha}^{\max}(X_i) = \mathbb{E}[\,X_i \,|\, X_{(d)} \geq Q_{X_{(d)}}(\alpha)\,]$ where $X_{(d)} = \max\{X_1, \ldots, X_d\}$

Landsman and Valdez (2003): elliptical distribution functions
Cai and Li (2005): phase-type distributions
Bargès, Cossette, Marceau (2009): Farlie-Gumbel-Morgenstern copulas

Measures of Systemic Risks

- Systemic risk in an interconnected network of financial institutions
- $X = (X_1, \ldots, X_d)$ where X_j is the risk exposure of company j.
- $L = X_1 + \cdots + X_d$ represents the aggregated risk in the firm network
- The CoVaR associated with company i

$$\mathrm{CoVaR}_\alpha^i(X) = \mathrm{VaR}_\alpha\left(L \mid X_i \geq \mathrm{VaR}_\alpha(X_i)\right)$$

Adrian and Brunnermeier (2011), Mainik and Schaanning (2012)

- The MES is defined by

$$\mathrm{MES}_\alpha^i(X) = \mathbb{E}\left[X_i \mid L \geq \mathrm{VaR}_\alpha(L)\right]$$

Acharya *et al.* (2010), Brownlees and Engle (2012), Cai *et al.* (2013)

Measures for risk with heterogeneous characteristics

- Risks that cannot be aggregated together

- Two kinds of literature:

  1) Extension of existing axioms to a multivariate setting

  Jouini, Meddeb, Touzi (2004), Burgert and Rüschendorf (2006), Rüschendorf (2006), Cascos and Molchanov (2007), Hamel and Heyde (2010), Ekeland, Galichon, Henry (2012)

  2) Quantile-based risk measures

  Massé and Theodorescu (1994), Koltchindkii (1997), Embrechts and Puccetti (2006), Nappo and Spizzichino (2009), Prékopa (2010), Lee and Prékopa (2012)

Construction of Multivariate Risk Measures

$$\rho: \quad \mathbf{X} := (X_1, \ldots, X_d) \mapsto \begin{pmatrix} \rho^1[\mathbf{X}] \\ \vdots \\ \rho^d[\mathbf{X}] \end{pmatrix} \in \mathbb{R}^d_+,$$

Some desirable properties:

- Combine in a concise way information on both marginals and risks dependencies
- Compatible with univariate version when $d = 1$
- Easily computable for large class of multivariate distribution functions
- Consistent with usual invariance properties (Artzner et al.'s axioms)
- Consistent behavior with respect to risk perturbations

Multivariate *Value-at-Risk* based on quantile curves (Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009):

$$\partial \underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}^d_+ : F(\mathbf{x}) = \alpha\} \qquad \partial \overline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}^d_+ : \overline{F}(\mathbf{x}) = 1 - \alpha\}$$

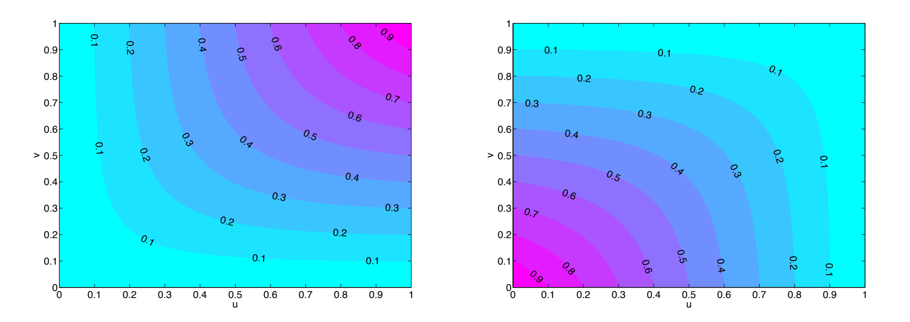


Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Value-at-Risk**

Definition

Consider a random vector $\mathbf{X}$ with absolutely continuous cdf F and survival function $\overline{F}$. For $\alpha \in (0,1)$, we define:

$$\underline{\mathrm{VaR}}_\alpha(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) = \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) = \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) = \alpha \,] \end{pmatrix}$$

$$\overline{\mathrm{VaR}}_\alpha(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \overline{F}(\mathbf{X}) = 1 - \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \overline{F}(\mathbf{X}) = 1 - \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \overline{F}(\mathbf{X}) = 1 - \alpha \,] \end{pmatrix}$$

When $d = 1$: $\underline{\mathrm{VaR}}_\alpha(X) = \overline{\mathrm{VaR}}_\alpha(X) = \mathrm{VaR}_\alpha(X)$

Invariance Properties

Proposition

Consider a risk portfolio $\boldsymbol{X} = (X_1, \ldots, X_d)$ *and a constant vector* $\boldsymbol{c} = (c_1, \ldots, c_d)$ *with positive components*

- *Positive Homogeneity:*

$$\underline{\mathrm{VaR}}_\alpha(c_1 X_1, \ldots, c_d X_d) = \left(c_1 \underline{\mathrm{VaR}}^1_\alpha(\boldsymbol{X}), \ldots, c_d \underline{\mathrm{VaR}}^d_\alpha(\boldsymbol{X})\right)^T$$

$$\overline{\mathrm{VaR}}_\alpha(c_1 X_1, \ldots, c_d X_d) = \left(c_1 \overline{\mathrm{VaR}}^1_\alpha(\boldsymbol{X}), \ldots, c_d \overline{\mathrm{VaR}}^d_\alpha(\boldsymbol{X})\right)^T$$

- *Translation Invariance:*

$$\underline{\mathrm{VaR}}_\alpha(\boldsymbol{c} + \boldsymbol{X}) = \boldsymbol{c} + \underline{\mathrm{VaR}}_\alpha(\boldsymbol{X}), \quad \overline{\mathrm{VaR}}_\alpha(\boldsymbol{c} + \boldsymbol{X}) = \boldsymbol{c} + \overline{\mathrm{VaR}}_\alpha(\boldsymbol{X})$$

Comonotonic additivity

Definition (π-comonotonicity: Puccetti and Scarsini (2010))

A couple $(\mathbf{X}, \mathbf{Y})$ of d-dimensional random vectors is said to be π-comonotonic if there exists a d-dimensional random vector $\mathbf{Z} = (Z_1, \dots, Z_d)$ and non-decreasing functions $f_1, \dots, f_d$, $g_1, \dots, g_d$ such that

$$(\mathbf{X}, \mathbf{Y}) \stackrel{d}{=} ((f_1(Z_1), \dots, f_d(Z_d)), (g_1(Z_1), \dots, g_d(Z_d)))$$

Proposition

Let $(\mathbf{X}, \mathbf{Y})$ be a π-comonotonic couple of random vectors, then

$$\underline{\mathrm{VaR}}_\alpha(\mathbf{X} + \mathbf{Y}) = \underline{\mathrm{VaR}}_\alpha(\mathbf{X}) + \underline{\mathrm{VaR}}_\alpha(\mathbf{Y}),$$

$$\overline{\mathrm{VaR}}_\alpha(\mathbf{X} + \mathbf{Y}) = \overline{\mathrm{VaR}}_\alpha(\mathbf{X}) + \overline{\mathrm{VaR}}_\alpha(\mathbf{Y})$$

Archimedean copula dependence structure

Proposition

Let $\mathbf{X}$ be a d-dimensional portfolio of risks with marginal distributions $F_1, \dots, F_d$.

- *If $\mathbf{X}$ admits an Archimedian copula with generator ϕ, then*

$$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = \mathbb{E}\left[F_i^{-1}\left(\phi^{-1}(S\phi(\alpha))\right)\right], \quad i = 1, \dots, d$$

- *If $\tilde{\mathbf{X}}$ admits an Archimedian survival copula with generator ϕ, then*

$$\overline{\mathrm{VaR}}^i_\alpha(\tilde{\mathbf{X}}) = \mathbb{E}\left[F_i^{-1}\left(1 - \phi^{-1}(S\phi(1-\alpha))\right)\right], \quad i = 1, \dots, d$$

where S is a random variable with $\mathrm{Beta}(1, d-1)$ distribution.

Proof : follows from McNeil and Nešlehová (2009) representation of Archimedean copulas

Archimedean copula dependence structure

Proposition (McNeil and Nešlehová (2009))

Let $\boldsymbol{U} = (U_1, \ldots, U_d)$ be distributed according to a d-dimensional Archimedian copula C with generator ϕ, then

$$(\phi(U_1), \ldots, \phi(U_d)) \stackrel{d}{=} R\boldsymbol{S},$$

where

- $\boldsymbol{S} = (S_1, \ldots, S_d)$ *is uniformly distributed on the unit simplex* $\left\{\boldsymbol{x} \geq 0 \mid \sum_{k=1}^{d} x_k = 1\right\}$
- R *is an independent non-negative random variable (radial part of* $(\phi(U_1), \ldots, \phi(U_d))$*)*

One can deduce that

1) $R \stackrel{d}{=} \phi(C(\mathbf{U}))$

2) $[\mathbf{U} \mid C(\mathbf{U}) = \alpha] \stackrel{d}{=} \left(\phi^{-1}\left(S_1\phi\left(\alpha\right)\right), \ldots, \phi^{-1}\left(S_d\phi\left(\alpha\right)\right)\right)$

Explicit expressions for Archimedean copulas

Proposition

- *If $\mathbf{X}$ is distributed as an* Archimedian copula *with generator ϕ, then*

$$\underline{\mathrm{VaR}}^i_\alpha(\mathbf{X}) = 1 - \int_\alpha^1 \left(1 - \frac{\phi(u)}{\phi(\alpha)}\right)^{d-1} du$$

- *If $\tilde{\mathbf{X}}$ has uniform marginal and admits an* Archimedian survival copula *with generator ϕ, i.e. $\tilde{\mathbf{X}} \overset{d}{=} 1 - \mathbf{X}$, then*

$$\overline{\mathrm{VaR}}^i_\alpha(\tilde{\mathbf{X}}) = \int_{1-\alpha}^1 \left(1 - \frac{\phi(u)}{\phi(1-\alpha)}\right)^{d-1} du$$

Explicit expressions for bivariate Clayton copulas

Generator ϕ of a Clayton copula with dependence parameter θ:

$$\phi(u) = \frac{1}{\theta}\left(u^{-\theta} - 1\right), \quad u \in (0,1)$$

Copula	θ	$\underline{\mathrm{VaR}}^{i}_{\alpha,\theta}(X,Y)$	$\overline{\mathrm{VaR}}^{i}_{\alpha,\theta}(\tilde{X},\tilde{Y})$
Clayton C_θ	$(-1,\infty)$	$\frac{\theta}{\theta-1}\frac{\alpha^\theta-\alpha}{\alpha^\theta-1}$	$1-\frac{\theta}{\theta-1}\frac{(1-\alpha)^\theta-(1-\alpha)}{(1-\alpha)^\theta-1}$
Counter-monotonic	-1	$\frac{1+\alpha}{2}$	$\frac{\alpha}{2}$
Independent	0	$\frac{\alpha-1}{\ln\alpha}$	$1+\frac{\alpha}{\ln(1-\alpha)}$
Comonotonic	∞	α	α

Table: Components $i = 1,2$ of $\underline{\mathrm{VaR}}^{i}$ and $\overline{\mathrm{VaR}}^{i}$ where (X,Y) follows a Clayton copula and $(\tilde{X},\tilde{Y}) := (1-X, 1-Y)$, i.e., $(\tilde{X},\tilde{Y})$ has a survival Clayton copula with uniform margins

Behavior of VaR components: bivariate Clayton case

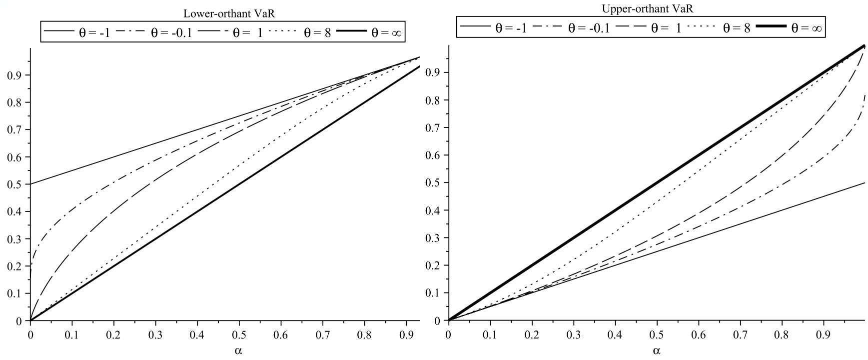


Figure: Behavior of $\underline{\mathrm{VaR}}^{1}_{\alpha,\theta}(X,Y)$ (**left**) and $\overline{\mathrm{VaR}}^{1}_{\alpha,\theta}(\tilde{X},\tilde{Y})$ (**right**) with respect to risk level α and dependence parameter θ

Comparison with univariate VaR, behavior with respect to α

	$\underline{\mathrm{VaR}}_\alpha(\mathbf{X})$	$\overline{\mathrm{VaR}}_\alpha(\mathbf{X})$
Comparison with univariate VaR	Univariate VaR is a lower bound: • $\underline{\mathrm{VaR}}^{i}_\alpha(\mathbf{X}) \geq \mathrm{VaR}_\alpha(X_i),\ \forall\, \alpha \in (0,1)$. Comonotonic case: • $\underline{\mathrm{VaR}}^{i}_\alpha(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i),\ \forall\, \alpha \in (0,1)$.	Univariate VaR is an upper bound: • $\overline{\mathrm{VaR}}^{i}_\alpha(\mathbf{X}) \leq \mathrm{VaR}_\alpha(X_i),\ \forall\, \alpha \in (0,1)$. Comonotonic case: • $\overline{\mathrm{VaR}}^{i}_\alpha(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i),\ \forall\, \alpha \in (0,1)$.
Risk level	$\underline{\mathrm{VaR}}^{i}_\alpha(\mathbf{X})$ is a non-decreasing function of α. ($\mathbf{X}$ with Archimedean copulas)	$\overline{\mathrm{VaR}}^{i}_\alpha(\tilde{\mathbf{X}})$ is a non-decreasing function of α. ($\tilde{\mathbf{X}}$ with Archimedean survival copulas)

Effect of risk perturbations

	$\underline{\mathrm{VaR}}_{\alpha}(\mathbf{X})$	$\overline{\mathrm{VaR}}_{\alpha}(\mathbf{X})$
Change in marginals	For a fixed copula C, if $X_i \overset{d}{=} Y_i$: • $\underline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{X}) = \underline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. For a fixed copula C, if $X_i \preceq_{st} Y_i$: • $\underline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{X}) \leq \underline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$.	For a fixed copula C, if $X_i \overset{d}{=} Y_i$: • $\overline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{X}) = \overline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. For a fixed copula C, if $X_i \preceq_{st} Y_i$: • $\overline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{X}) \leq \overline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$.
Change in dependence structure	For fixed marginals, if $\theta \leq \theta^*$: • $\underline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{X}) \leq \underline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. $\mathbf{X}$ with Archimedean copula	For fixed marginals, if $\theta \leq \theta^*$: • $\overline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{X}) \leq \overline{\mathrm{VaR}}^{i}_{\alpha}(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. $\tilde{\mathbf{X}}$ with Archimedean survival copula

Multivariate *CTE*-s based on upper-level set of multivariate cdf and lower-level set of survival functions:

$$\underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) \geq \alpha\} \qquad \overline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : \overline{F}(\mathbf{x}) \leq 1 - \alpha\}$$

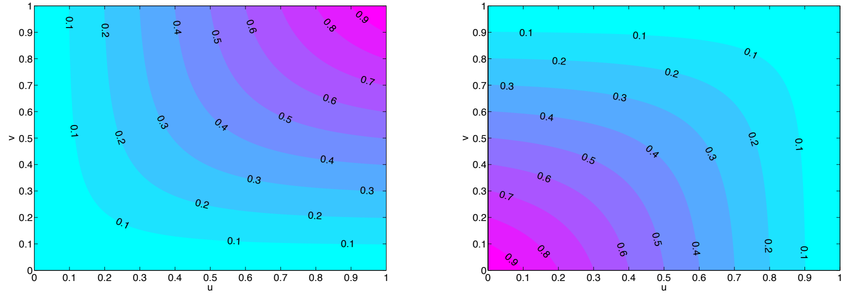


Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Conditional-Tail-Expectation**

Definition

Consider a random vector $\mathbf{X}$ with absolutely continuous cdf F and survival function $\overline{F}$. For $\alpha \in (0,1)$, we define:

$$\underline{\mathrm{CTE}}_\alpha(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) \geq \alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) \geq \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) \geq \alpha \,] \end{pmatrix}$$

$$\overline{\mathrm{CTE}}_\alpha(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \overline{F}(\mathbf{X}) \leq 1-\alpha] = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \overline{F}(\mathbf{X}) \leq 1-\alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \overline{F}(\mathbf{X}) \leq 1-\alpha \,] \end{pmatrix}$$

When $d = 1$: $\underline{\mathrm{CTE}}_\alpha(X) = \overline{\mathrm{CTE}}_\alpha(X) = \mathrm{CTE}_\alpha(X)$

Invariance Properties

- Positive Homogeneity: $\forall\ \mathbf{c} \in \mathbb{R}_+^d$,

$$\underline{\mathrm{CTE}}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\underline{\mathrm{CTE}}_\alpha(\mathbf{X}),\ \overline{\mathrm{CTE}}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\overline{\mathrm{CTE}}_\alpha(\mathbf{X})$$

- Translation Invariance: $\forall\ \mathbf{c} \in \mathbb{R}_+^d$,

$$\underline{\mathrm{CTE}}_\alpha(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \underline{\mathrm{CTE}}_\alpha(\mathbf{X}),\ \overline{\mathrm{CTE}}_\alpha(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \overline{\mathrm{CTE}}_\alpha(\mathbf{X})$$

- π-comonotonic additivity: if $(\mathbf{X}, \mathbf{Y})$ is π-comonotonic, then

$$\underline{\mathrm{CTE}}_\alpha(\mathbf{X} + \mathbf{Y}) = \underline{\mathrm{CTE}}_\alpha(\mathbf{X}) + \underline{\mathrm{CTE}}_\alpha(\mathbf{Y}),$$

$$\overline{\mathrm{CTE}}_\alpha(\mathbf{X} + \mathbf{Y}) = \overline{\mathrm{CTE}}_\alpha(\mathbf{X}) + \overline{\mathrm{CTE}}_\alpha(\mathbf{Y})$$

- For $\alpha = 0$, $\underline{\mathrm{CTE}}_0(\mathbf{X}) = \overline{\mathrm{CTE}}_0(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$

Link with multivariate VaR-s

Proposition

Under the regularity assumption, the following integral representations hold

$$\underline{CTE}^i_\alpha(\boldsymbol{X}) = \frac{1}{1 - K(\alpha)} \int_\alpha^1 \underline{VaR}^i_\gamma(\boldsymbol{X}) K'(\gamma) d\gamma,$$

$$\overline{CTE}^i_\alpha(\boldsymbol{X}) = \frac{1}{\widehat{K}(1 - \alpha)} \int_\alpha^1 \overline{VaR}^i_\gamma(\boldsymbol{X}) \widehat{K}'(1 - \gamma) d\gamma,$$

where

- K *is the Kendall distribution of* $\boldsymbol{X}$*:* $K(x) = \mathbb{P}\left(F(\boldsymbol{X}) \leq x\right)$, *for all* x *in* $(0, 1)$
- $\widehat{K}$ *is the upper-orthant Kendall distribution of* $\boldsymbol{X}$*:* $\widehat{K}(x) = \mathbb{P}\left(\overline{F}(\boldsymbol{X}) \leq x\right)$, *for all* x *in* $(0, 1)$.

Explicit expressions for bivariate Clayton copulas

Copula	θ	$\underline{\mathrm{CTE}}^{i}_{\alpha,\theta}(X, Y)$
Clayton C_θ	$(-1, \infty)$	$\frac{1}{2}\,\frac{\theta}{\theta-1}\,\frac{\theta-1-\alpha^{2}(1+\theta)+2\,\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic	-1	$\frac{1}{4}\,\frac{1-\alpha^{2}+2\ln\alpha}{1-\alpha+\ln\alpha}$
Independent	0	$\frac{1}{2}\,\frac{(1-\alpha)^{2}}{1-\alpha+\alpha\ln\alpha}$
Comonotonic	∞	$\frac{1+\alpha}{2}$

Table: Components $i = 1, 2$ of $\underline{\mathrm{CTE}}^{i}$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \underline{\mathrm{CTE}}^{i}_{\alpha,\theta}}{\partial\alpha} \geq 0$ and $\frac{\partial \underline{\mathrm{CTE}}^{i}_{\alpha,\theta}}{\partial\theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0, 1)$.

Behavior of CTE components: Clayton copula case

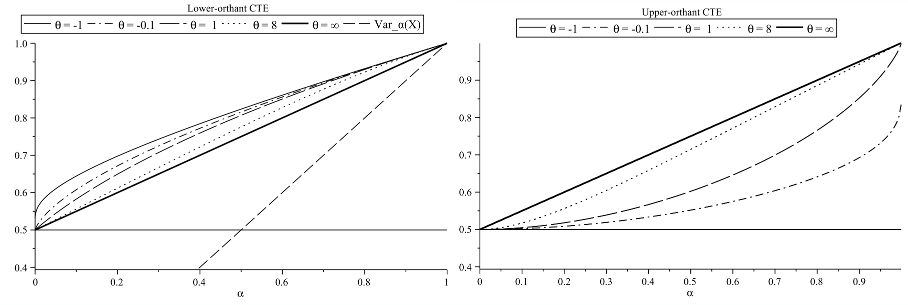


Figure: Behavior of $\underline{\mathrm{CTE}}^i_{\alpha,\theta}(X, Y)$ (left) and $\overline{\mathrm{CTE}}^i_{\alpha,\theta}(1-X, 1-Y)$ (right) with respect to risk level α for different values of dependence parameter θ.

Comonotonic case, behavior with respect to α, comparison with multivariate VaR

	$\underline{\mathrm{CTE}}_\alpha(\mathbf{X})$	$\overline{\mathrm{CTE}}_\alpha(\mathbf{X})$
	Comonotonic case: • $\underline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) = \mathrm{CTE}_\alpha(X_i), \ \forall\, \alpha \in (0,1)$.	Comonotonic case: • $\overline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) = \mathrm{CTE}_\alpha(X_i), \ \forall\, \alpha \in (0,1)$.
Risk level	• $\underline{\mathrm{CTE}}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α. ($\mathbf{X}$ with Archimedean copulas) • $\underline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) \geq \underline{\mathrm{VaR}}^i_\alpha(\mathbf{X})$, for all $\alpha \in (0,1)$.	• $\overline{\mathrm{CTE}}^i_\alpha(\tilde{\mathbf{X}})$ is a non-decreasing function of α. ($\tilde{\mathbf{X}}$ with Archimedean survival copulas) • $\overline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) \geq \overline{\mathrm{VaR}}^i_\alpha(\mathbf{X})$, for all $\alpha \in (0,1)$.

Effect of risk perturbations

	$\underline{CTE}_\alpha(\mathbf{X})$	$\overline{CTE}_\alpha(\mathbf{X})$
Change in marginals	For a fixed copula C, if $X_i \overset{d}{=} Y_i$: • $\underline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) = \underline{\mathrm{CTE}}^i_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. For a fixed copula C, if $X_i \preceq_{st} Y_i$: • $\underline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) \leq \underline{\mathrm{CTE}}^i_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$.	For a fixed copula C, if $X_i \overset{d}{=} Y_i$: • $\overline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) = \overline{\mathrm{CTE}}^i_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. For a fixed copula C, if $X_i \preceq_{st} Y_i$: • $\overline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) \leq \overline{\mathrm{CTE}}^i_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$.
Change in dependence structure	For fixed marginals, if $\theta \leq \theta^*$: • $\underline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) \leq \underline{\mathrm{CTE}}^i_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. $\mathbf{X}$ with Archimedean copula	For fixed marginals, if $\theta \leq \theta^*$: • $\overline{\mathrm{CTE}}^i_\alpha(\mathbf{X}) \leq \overline{\mathrm{CTE}}^i_\alpha(\mathbf{Y}),\ \forall\, \alpha \in (0,1)$. $\tilde{\mathbf{X}}$ with Archimedean survival copula

Perspectives

- Subadditivity of the proposed multivariate CTE-s ?
- Allocation problems for portfolios of portfolios
- Other perspectives:
  Extension to discrete distribution functions
  Comparison with other multivariate risk measures

Thank you for your attention

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