Hedging default risks of CDOs in Markovian contagion models

Areski COUSIN

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Introduction

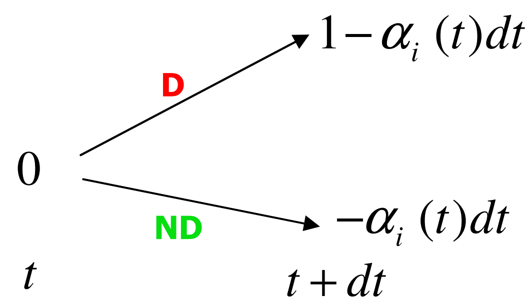
- Purpose of the paper
  - Describe a hedging strategy of CDO tranches
  - Based upon dynamic trading of corresponding CDS Index and the risk-free asset
- Contagion models
  - Class of intensity models …
  - Credit spreads only depend on the history of default events
    - Credit spreads are deterministic between two default dates
    - Default Risk governs Credit Spread Risk
- Homogeneous credit portfolio
  - No individual name effect
  - Only need of the CDS Index
- Markovian dynamics of default intensities
  - Pricing and hedging CDO within a binomial tree

Introduction

- Dynamic hedging of defaultable contingent claim in **complete market**
  - Blanchet-Scaillet & Jeanblanc [2004]
- Dynamic hedging of basket credit derivatives in **complete market**
  - Bielecki, Jeanblanc & Rutkowski [2007], Frey & Backhaus [2006]
- Dynamic hedging in **asymptotically complete market**
  - Laurent [2006]
- Dynamic hedging in **incomplete market**
  - Super-replication : Walker [2005]
  - Quadratic hedging : Becherer & Schweizer [2005], Elouerkhaoui [2006]

Martingale Representation Theorem

- Some notations :
  - $\tau_1, \ldots, \tau_n$: default dates of counterparties 1,…,n
  - H_t : natural filtration of default dates
  - $N_1(t) = 1_{\{\tau_1 \le t\}}, \ldots, N_n(t) = 1_{\{\tau_n \le t\}}$: default indicators at date t
  - $N(t) = \sum_{i=1}^{n} N_i(t)$: number of default at date t
  - $\alpha_1(t), \ldots, \alpha_n(t)$: spreads of **instantaneous CDS**
  - Probabillity Q such that
    - under Q, $\alpha_1(t), \ldots, \alpha_n(t)$ are default intensities of $N_1(t), \ldots, N_n(t)$



Martingale Representation Theorem

- Integral representation of point process martingale
  - Jacod [1975], Brémaud Chap. III
  - **No simultaneous default**

$$M = E^{Q}\left[M\right] + \sum_{i=1}^{n} \int_{0}^{T} \theta_i(s)\left(dN_i(s) - \alpha_i(s)ds\right)$$

- M : H_T-mesurable Q-integrable payoff
  - CDO Tranches payoff can be perfecty replicated
  - Using n instantaneous CDS

⇒ **Does not provide a practical way to construct hedging stategies**

Markovian homogeneous contagion model

- Contagion models : Davis & Lo[2001], Jarrow & Yu[2001], Yu[2001]
  - Default intensities depend on the complete history of defaults

$$Q\left(\tau_i \in [t, t+dt] \,\middle|\, H_t\right) = \alpha_i\left(t, H_t\right) dt, \;\; i = 1, \ldots, n$$

- Homogeneous assumption
  - Default intensities are the same for all names ⟹ α
  - Total loss is simply expressed as $L(t) = (1-R)\dfrac{N(t)}{n}$

Recovery rate

- Homogeneous + Markovian assumption
  - Default intensities only depend on the current number of defaults

$$Q\left(\tau_i \in [t, t+dt] \,\middle|\, H_t\right) = Q\left(\tau_i \in [t, t+dt] \,\middle|\, N_t\right) = \alpha\left(t, N(t)\right) dt, \;\; i = 1, \ldots, n$$

Markovian homogeneous contagion model

- No simultaneous defaults assumption
  - Intensity λ of the number of defaults process $N(t)$ is simply the sum of individual default intensities:

$$\lambda(t, N(t)) = (n - N(t)) \times \alpha(t, N(t))$$

  - The process $N(t)$ is a Markov chain (a pure death process) with generator :

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 & & & 0 \\ 0 & & \bullet & \bullet & & & 0 \\ 0 & & & & \bullet & & 0 \\ 0 & & & & & \bullet & 0 \\ 0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

  - $\{N(t) = n\}$ is an absorbing state

Tree Approach to hedging defaults

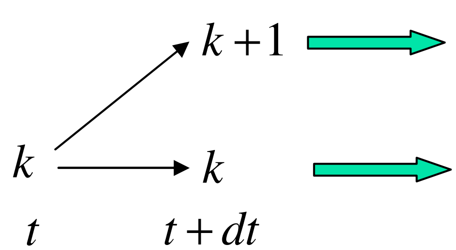
- Computation of Index and CDO tranche premiums
  - Based on the distribution of the aggregated loss $L(t) = (1-R)\frac{N(t)}{n}$

- The transition matrix of $N(t)$ can be expressed as

$$Q\left(t,t'\right) = \exp\left(\int_t^{t'} \Lambda(s)ds\right)$$

  - Arnsdorf & Halperin[2007], Herbertsson[2007]

- Suppose that k defaults have occured at time t :

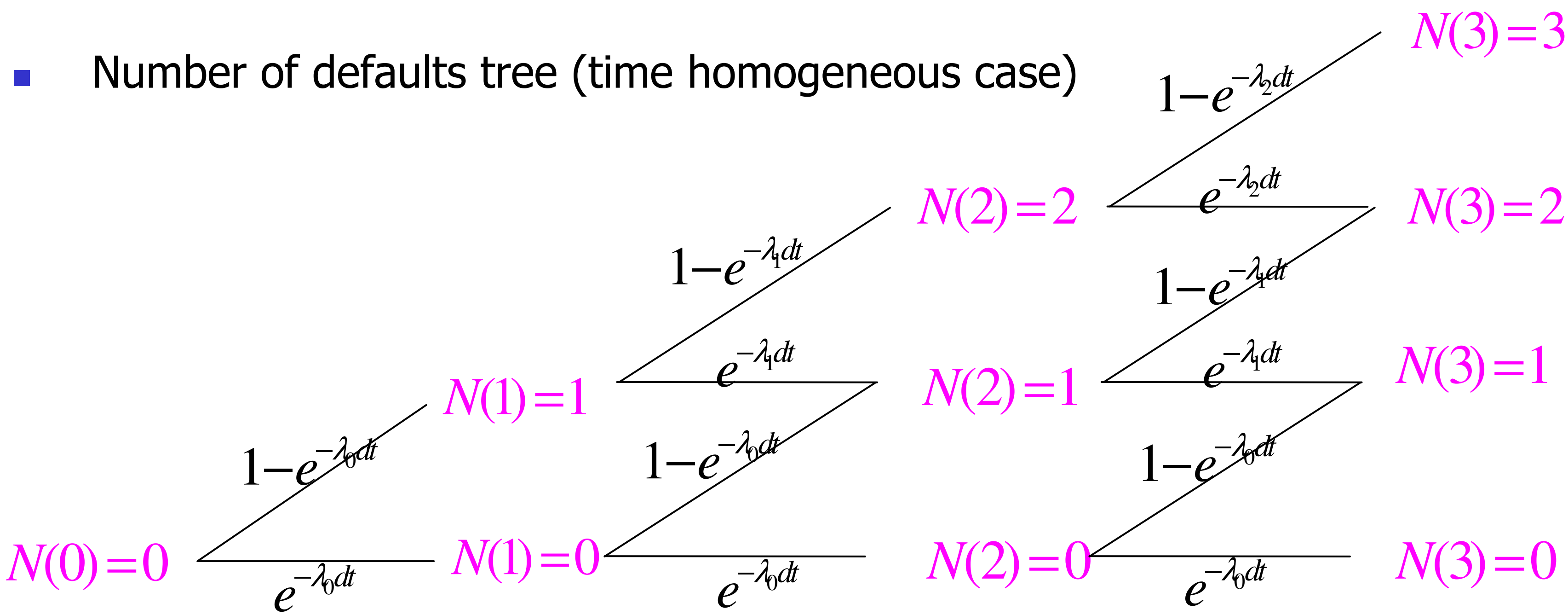


$$Q\left(N(t+dt) = k+1 \middle| N(t) = k\right) \approx 1 - \mathrm{e}^{-\lambda(t,k)dt}$$

$$Q\left(N(t+dt) = k \middle| N(t) = k\right) \approx \mathrm{e}^{-\lambda(t,k)dt}$$

Tree Approach to hedging defaults

- Number of defaults tree (time homogeneous case)

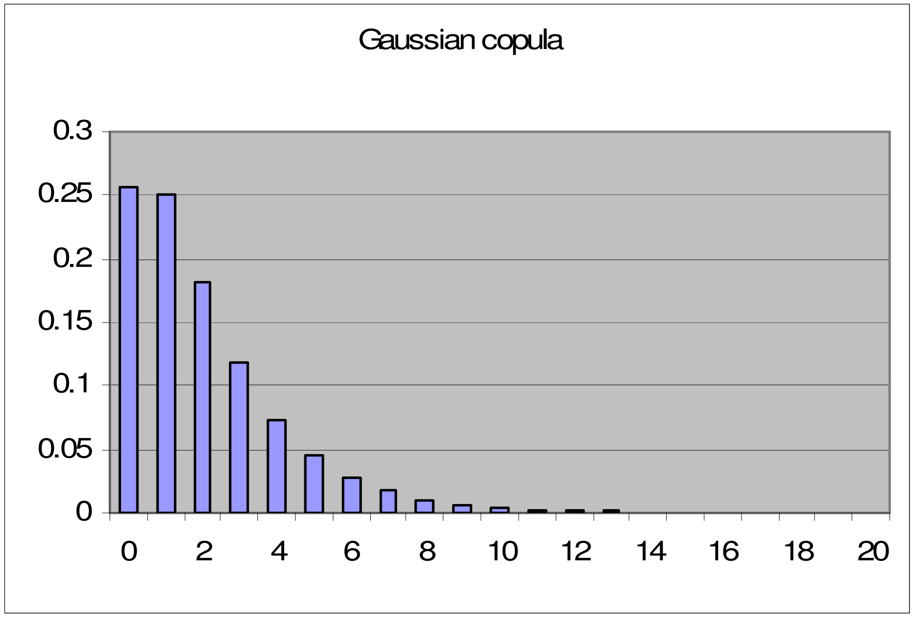


- Calibration of $\lambda_0, \dots \lambda_n$ on marginal distribution of $N(t)$
  - forward induction
- Computation of CDO Tranches and Index present values
  - backward induction

Computation of deltas

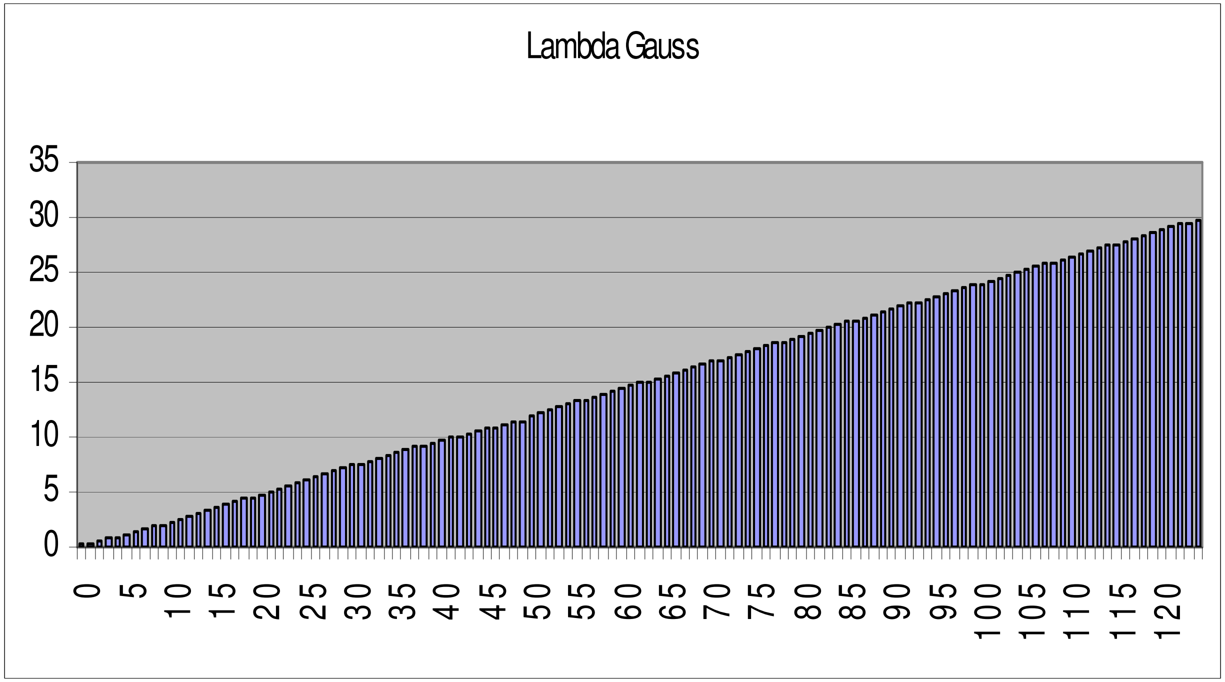
- Calibration of loss intensities $\lambda_0, \ldots \lambda_n$ on a gaussian copula distribution

  - Homogeneous portfolio $n = 125$
  - T = 5 years
  - CDS Spreads : 20 bps per annum
  - Recovery rate $R = 40\%$
  - Correlation $\rho = 30\%$
  - $Q(N(t) = k),\ k = 0, \ldots, 20$



Computation of deltas

- Calibration of loss intensities $\lambda_0, \ldots \lambda_n$ on a gaussian copula distribution
  - Figure below represents loss intensities, with respect to the number of defaults
  - Increase in intensities: contagion effects



Computation of deltas

- Dynamics of credit deltas: $\delta(t,k) = \dfrac{CDO(t+1,k+1) - CDO(t+1,k)}{Index(t+1,k+1) - Index(t+1,k)}$

- Credit deltas - Tranche equity [0,3%]

		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
Nb Defaults	0	3.00%	0.810	0.839	0.865	0.889	0.911	0.929	0.946
	1	2.52%	0	0.613	0.657	0.701	0.743	0.785	0.823
	2	2.04%	0	0.343	0.386	0.432	0.483	0.536	0.591
	3	1.56%	0	0.142	0.167	0.197	0.231	0.271	0.318
	4	1.08%	0	0.046	0.055	0.066	0.080	0.097	0.119
	5	0.60%	0	0.014	0.015	0.018	0.021	0.025	0.031
	6	0.12%	0	0.002	0.002	0.002	0.003	0.003	0.004
	7	0.00%	0	0	0	0	0	0	0

- Gradually decrease with the number of defaults
  - concave payoff
  - When the number of default is > 6, the tranche is exhausted, delta = 0
- Credit deltas increase with time

Computation of deltas

- Credit deltas - Tranche [3,6%]

Nb Defaults	OutStanding Nominal	Weeks 0	14	28	42	56	70	84
0	3.00%	0.162	0.139	0.118	0.097	0.078	0.061	0.046
1	3.00%	0	0.325	0.296	0.265	0.232	0.198	0.164
2	3.00%	0	0.492	0.484	0.468	0.444	0.413	0.374
3	3.00%	0	0.516	0.546	0.570	0.584	0.588	0.580
4	3.00%	0	0.399	0.451	0.505	0.556	0.604	0.645
5	3.00%	0	0.242	0.289	0.344	0.405	0.471	0.540
6	3.00%	0	0.126	0.156	0.193	0.238	0.293	0.359
7	2.64%	0	0.061	0.075	0.093	0.118	0.150	0.193
8	2.16%	0	0.032	0.037	0.044	0.054	0.068	0.089
9	1.68%	0	0.019	0.021	0.023	0.027	0.032	0.039
10	1.20%	0	0.012	0.012	0.013	0.015	0.016	0.018
11	0.72%	0	0.006	0.007	0.007	0.008	0.008	0.009
12	0.24%	0	0.002	0.002	0.002	0.002	0.002	0.003
13	0.00%	0	0	0	0	0	0	0

- When the number of default is > 12, the tranche is exhausted

Conclusion

- Thanks to stringent assumptions
  - Credit spreads driven by defaults
  - Homogeneity
  - Markov property
- It is possible to compute a dynamic hedging strategy
  - Based on the CDS Index
- That fully replicates the CDO tranche payoffs
  - Very simple implementation using a recombining tree
- Credit spread dynamics need to be improved