Hedging default risk of CDOs in Markovian contagion models

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26 Novembre 2007

Presentation related to paper:

Hedging default risk of CDOs in Markovian contagion models (2007)

Joint work with Jean-Paul Laurent and Jean-David Fermanian

Available on www.defaultrisk.com

Introduction

- In interest rate or equity markets, pricing is related to the cost of the hedge
  - ex: Blask-Scholes pricing model of equity options
- In credit markets, pricing is disconnect from hedging
  - ex: The industrial CDO pricing model, use of local hedging strategies
- Need to relate pricing and hedging
- In defaultrisk.com
  - More than 1000 papers
  - About 10 papers deal with hedging issues
    - **Among others, Bielecki, Jeanblanc & Rutkowski (2007), Frey & Backhaus (2007), Laurent (2006)**

Introduction

- Purpose of the presentation
  - Not trying to embrace all risk management issues
  - Focus on very specific aspects of default and credit spread risk
  - Obtain replication strategies for CDO tranches
- Overlook of the presentation
  - Tree approach to hedging defaults
  - Results and comments

Tree approach to hedging defaults

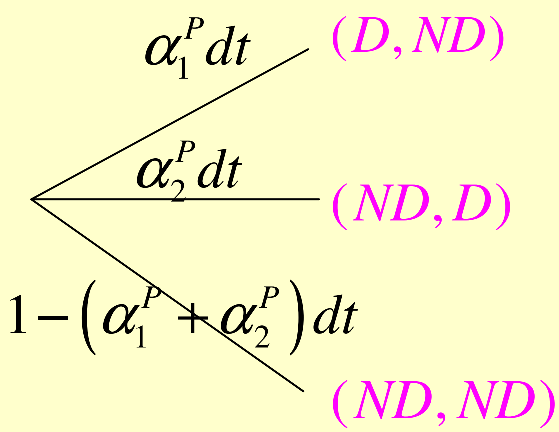
- We will start with two names only
- Firstly in a static framework
  - Hedging a First to Default Swap
  - Discuss historical and risk-neutral probabilities
- Further extending the model to a dynamic framework
  - Computation of prices and hedging strategies along the tree
  - Pricing and hedging of zero coupon CDO tranches
- Multiname case: homogeneous Markovian model
  - Computation of risk-neutral tree for the loss
  - Computation of dynamic deltas
- Technical details can be found in the paper:
  - "hedging default risks of CDOs in Markovian contagion models"

Tree approach to hedging defaults

- Some notations :
  - τ_1, τ_2 default times of counterparties 1 and 2,
  - $\mathcal{H}_t$ available information at time t,
  - P historical probability,
  - α_1^P, α_2^P : (historical) default intensities:
    - $P\left[\tau_i \in [t, t+dt[\,\middle|\, H_t\right] = \alpha_i^P dt,\ i = 1,2$
- Assumption of « local » independence between default events
  - Probability of 1 and 2 defaulting altogether:
    - $P\left[\tau_1 \in [t, t+dt[, \tau_2 \in [t, t+dt[\,\middle|\, H_t\right] = \alpha_1^P dt \times \alpha_2^P dt \text{ in } (dt)^2$
  - Local independence: simultaneous joint defaults can be neglected

Tree approach to hedging defaults

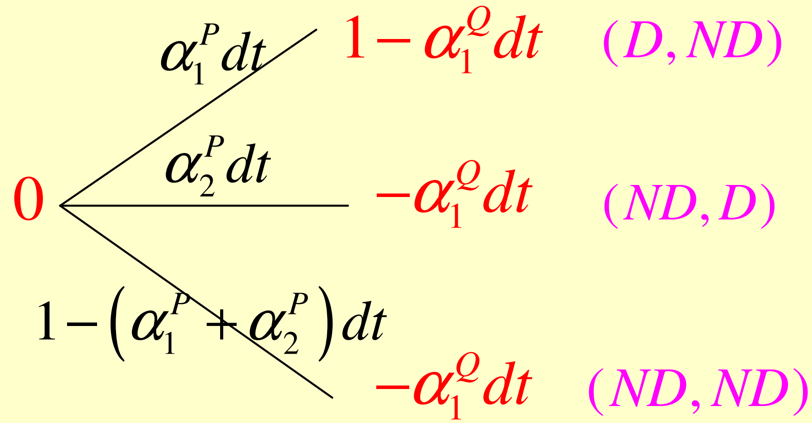
- Building up a tree:
  - Four possible states: (*D*,*D*), (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
  - Under no simultaneous defaults assumption $p_{(D,D)}=0$
  - Only three possible states: (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
  - Identifying (historical) tree probabilities:



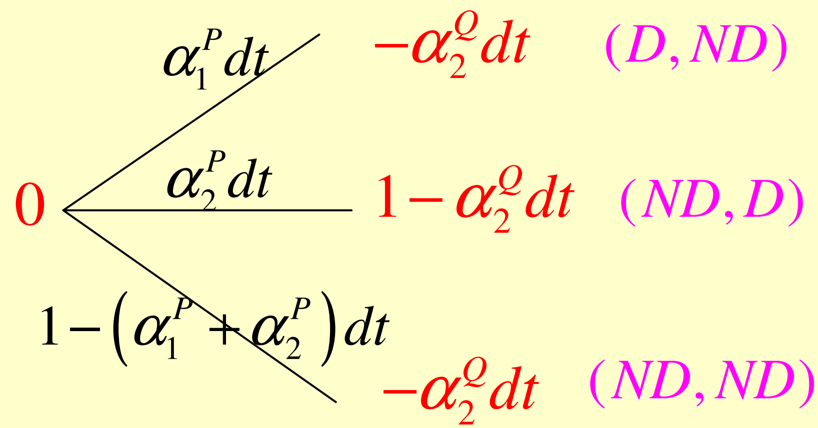
$$\begin{cases} p_{(D,D)}=0 \Rightarrow p_{(D,ND)}=p_{(D,D)}+p_{(D,ND)}=p_{(D,.)}=\alpha_1^P dt \\ p_{(D,D)}=0 \Rightarrow p_{(ND,D)}=p_{(D,D)}+p_{(ND,D)}=p_{(.,D)}=\alpha_2^P dt \\ p_{(ND,ND)}=1-p_{(D,.)}-p_{(.,D)} \end{cases}$$

Tree approach to hedging defaults

- Stylized cash flows of short term digital CDS on counterparty 1:
  - CDS 1 premium $\alpha_1^Q dt$



- Stylized cash flows of short term digital CDS on counterparty 2:



Tree approach to hedging defaults

- Cash flows of short term digital first to default swap with premium $\alpha_F^Q dt$:

  From 0:
  - $\alpha_1^P dt$: $1-\alpha_F^Q dt$ (D, ND)
  - $\alpha_2^P dt$: $1-\alpha_F^Q dt$ (ND, D)
  - $1-\left(\alpha_1^P+\alpha_2^P\right)dt$: $-\alpha_F^Q dt$ (ND, ND)

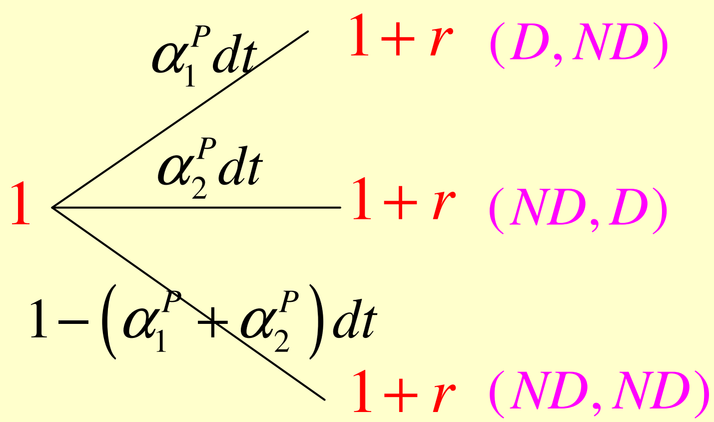
- Cash flows of holding CDS 1 + CDS 2:

  From 0:
  - $\alpha_1^P dt$: $1-\left(\alpha_1^Q+\alpha_2^Q\right)dt$ (D, ND)
  - $\alpha_2^P dt$: $1-\left(\alpha_1^Q+\alpha_2^Q\right)dt$ (ND, D)
  - $1-\left(\alpha_1^P+\alpha_2^P\right)dt$: $-\left(\alpha_1^Q+\alpha_2^Q\right)dt$ (ND, ND)

- Perfect hedge of first to default swap by holding 1 CDS 1 + 1 CDS 2
  - Delta with respect to CDS 1 = 1, delta with respect to CDS 2 = 1

Tree approach to hedging defaults

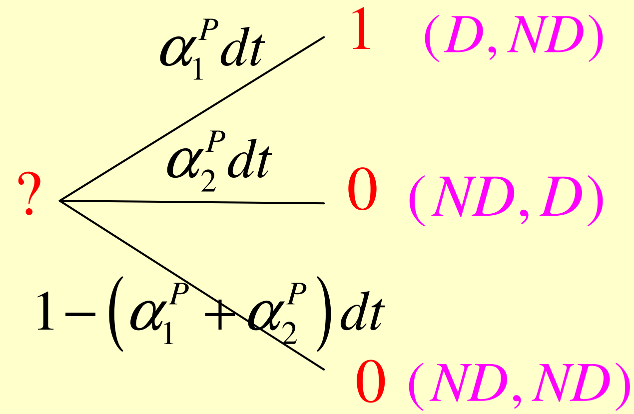
- Absence of arbitrage opportunities imply:
  - $\alpha_F^Q = \alpha_1^Q + \alpha_2^Q$
- Arbitrage free first to default swap premium
  - Does not depend on historical probabilities α_1^P, α_2^P
- Three possible states: (*D*,*ND*), (*ND*,*D*), (*ND*,*ND*)
- Three tradable assets: CDS1, CDS2, risk-free asset

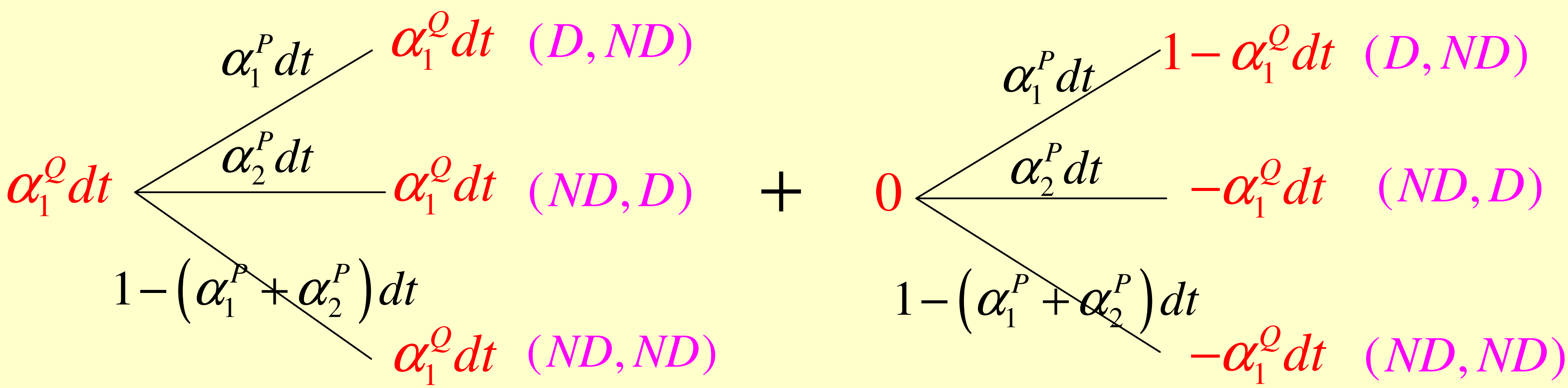


- For simplicity, let us assume $r = 0$

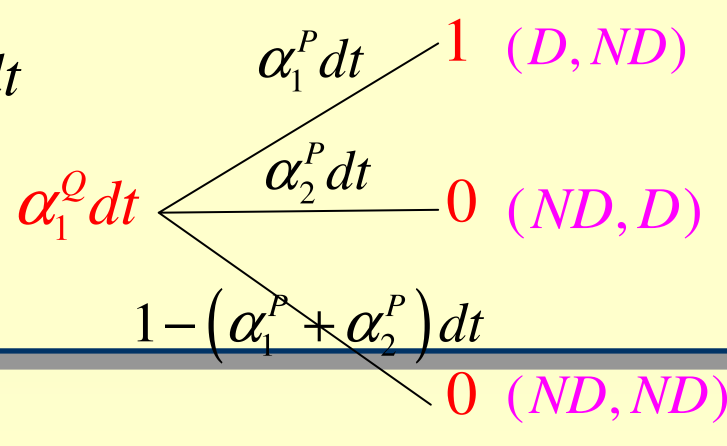
Tree approach to hedging defaults

- Three state contingent claims
  - Example: claim contingent on state (D, ND)
  - Can be replicated by holding
  - $\alpha_1^Q dt$ risk-free asset + 1 CDS 1



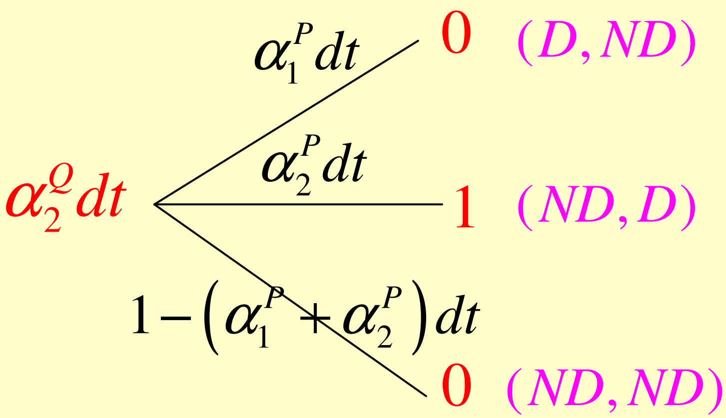


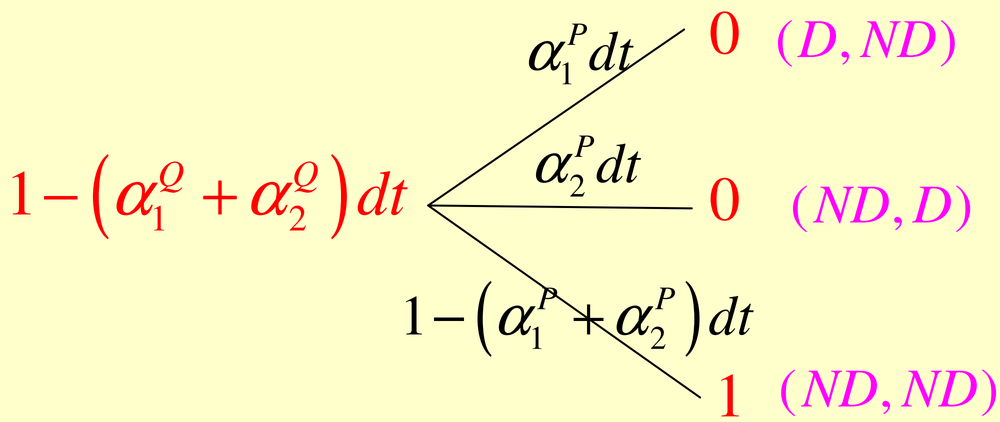
  - Replication price = $\alpha_1^Q dt$



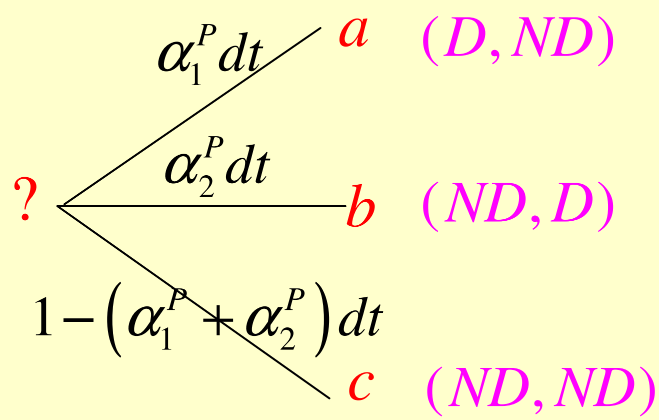
Tree approach to hedging defaults

- Similarly, the replication prices of the (ND, D) and (ND, ND) claims





- Replication price of:

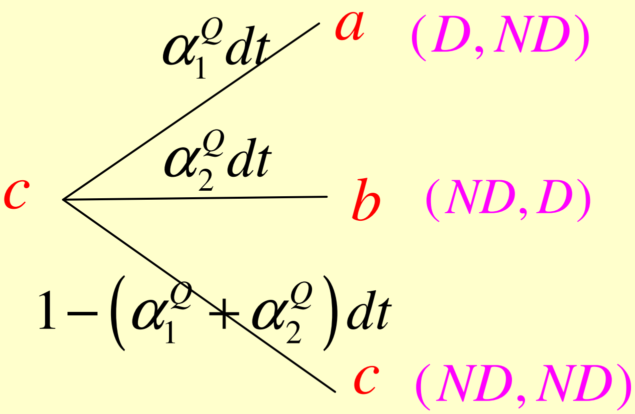


- Replication price = $a \times \alpha_1^Q dt + b \times \alpha_2^Q dt + c \times \left(1-(\alpha_1^Q+\alpha_2^Q)dt\right)$

Tree approach to hedging defaults

- Replication price obtained by computing the expected payoff
  - Along a risk-neutral tree

$$\alpha_1^Q dt \times a + \alpha_2^Q dt \times b + \left(1 - (\alpha_1^Q + \alpha_2^Q)dt\right)c$$



- Risk-neutral probabilities
  - Used for computing replication prices
  - Uniquely determined from short term CDS premiums
  - No need of historical default probabilities

Tree approach to hedging defaults

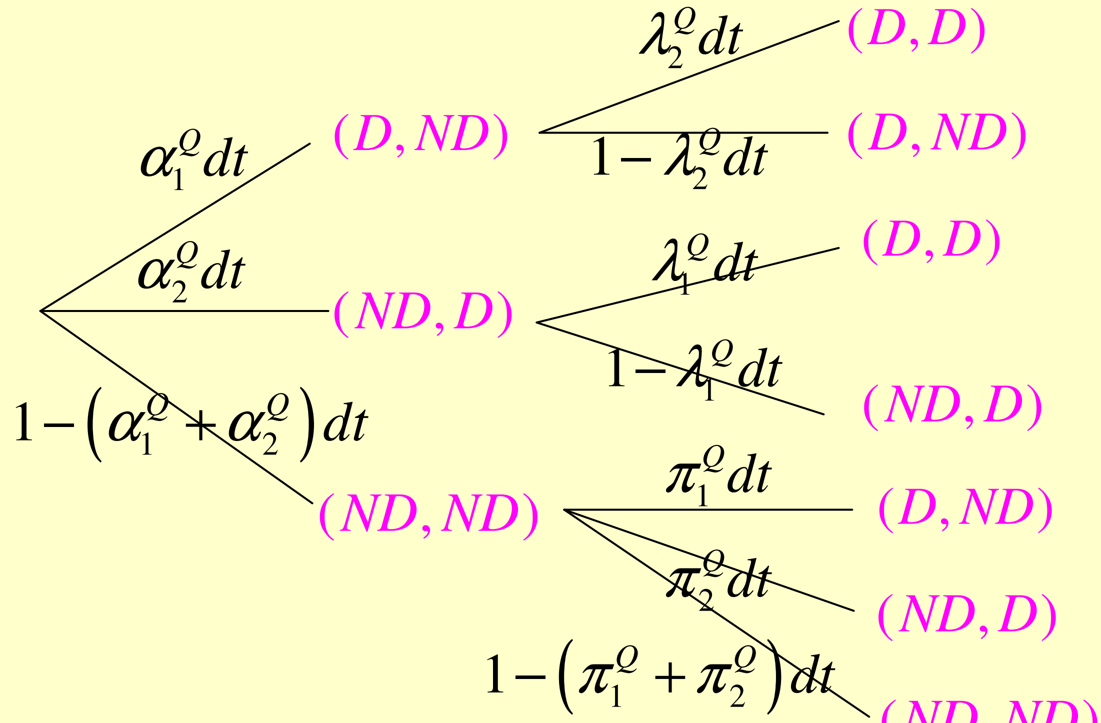
- Computation of deltas
  - Delta with respect to risk-free asset: δ_0
  - Delta with respect to CDS 1: δ_1
  - Delta with respect to CDS 2: δ_2

$$
\begin{cases}
a = \delta_0 + \delta_1 \times \overbrace{\left(1 - \alpha_1^Q dt\right)}^{\text{payoff CDS 1}} + \delta_2 \times \overbrace{\left(-\alpha_2^Q dt\right)}^{\text{payoff CDS 2}} \\
b = \delta_0 + \delta_1 \times \left(-\alpha_1^Q dt\right) + \delta_2 \times \left(1 - \alpha_2^Q dt\right) \\
c = \delta_0 + \delta_1 \times \underbrace{\left(-\alpha_1^Q dt\right)}_{\text{payoff CDS 1}} + \delta_2 \times \underbrace{\left(-\alpha_2^Q dt\right)}_{\text{payoff CDS 2}}
\end{cases}
$$

  - As for the replication price, deltas only depend upon CDS premiums

Tree approach to hedging defaults

- Dynamic case:



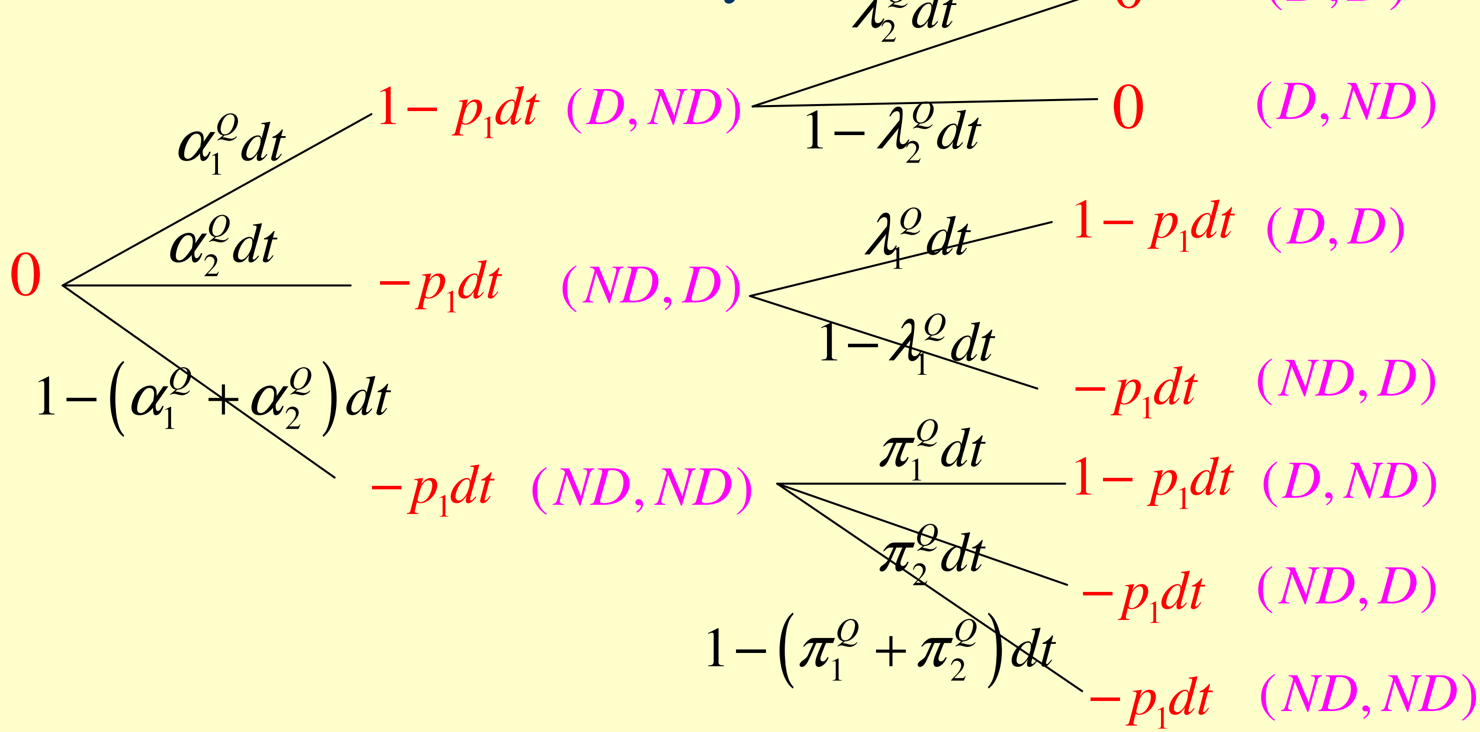
  - $\lambda_2^Q dt$ CDS 2 premium after default of name 1
  - $\lambda_1^Q dt$ CDS 1 premium after default of name 2
  - $\pi_1^Q dt$ CDS 1 premium if no name defaults at period 1
  - $\pi_2^Q dt$ CDS 2 premium if no name defaults at period 1
- Change in CDS premiums due to contagion effects
  - Usually, $\pi_1^Q < \alpha_1^Q < \lambda_1^Q$ and $\pi_2^Q < \alpha_2^Q < \lambda_2^Q$

Tree approach to hedging defaults

- Computation of prices and hedging strategies by backward induction
  - use of the dynamic risk-neutral tree
  - Start from period 2, compute price at period 1 for the three possible nodes
  - + hedge ratios in short term CDS 1,2 at period 1
  - Compute price and hedge ratio in short term CDS 1,2 at time 0
- Example to be detailed:
  - computation of CDS 1 premium, maturity = 2
  - $p_1 dt$ will denote the periodic premium
  - Cash-flow along the nodes of the tree

Tree approach to hedging defaults

- Computations CDS on name 1, maturity = 2

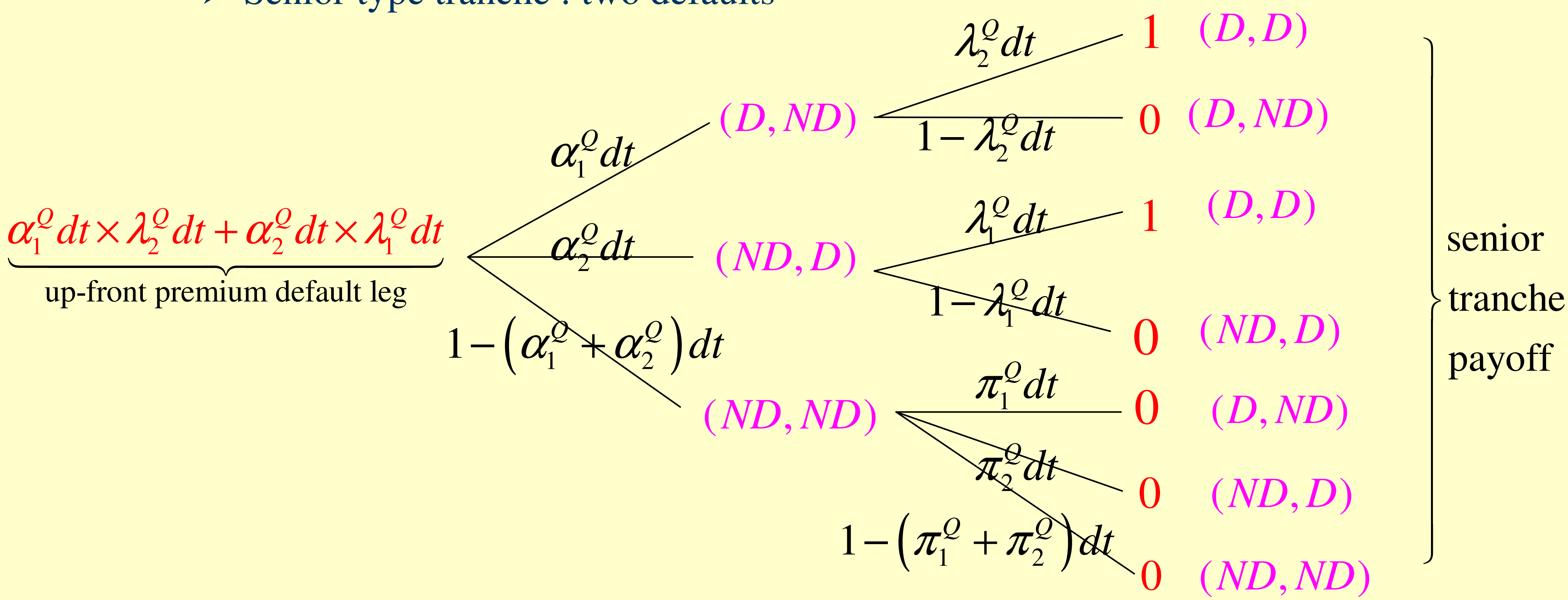


- Premium of CDS on name 1, maturity = 2, time = 0, $dt=1$, p_1 solves for:

$$0=\left(1-p_1\right)\alpha_1^Q+\left(-p_1+\left(1-p_1\right)\lambda_1^Q-p_1\left(1-\lambda_1^Q\right)\right)\alpha_2^Q$$
$$+\left(-p_1+\left(1-p_1\right)\pi_1^Q-p_1\pi_2^Q-p_1\left(1-\pi_1^Q-\pi_2^Q\right)\right)\left(1-\alpha_1^Q-\alpha_2^Q\right)$$

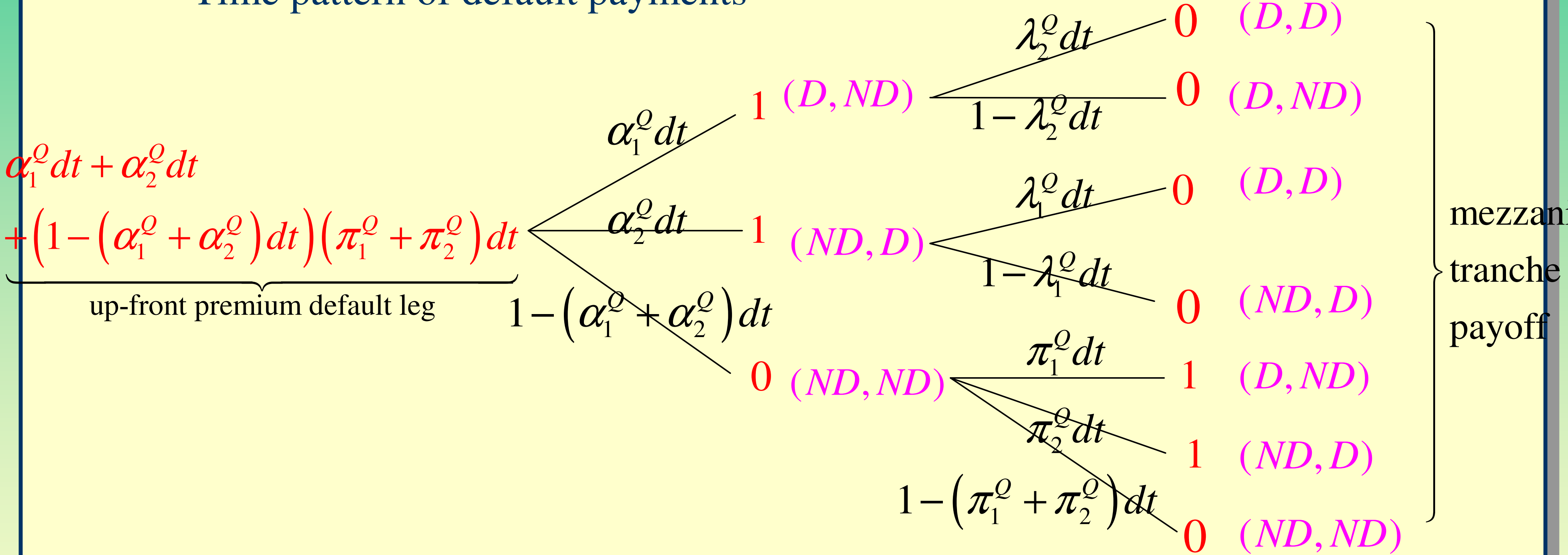
Tree approach to hedging defaults

- Example: stylized zero coupon CDO tranchelets
  - Zero-recovery, maturity 2
  - Aggregate loss at time 2 can be equal to 0,1,2
    - ➢ Equity type tranche contingent on no defaults
    - ➢ Mezzanine type tranche : one default
    - ➢ Senior type tranche : two defaults



Tree approach to hedging defaults

- mezzanine tranche
  - Time pattern of default payments



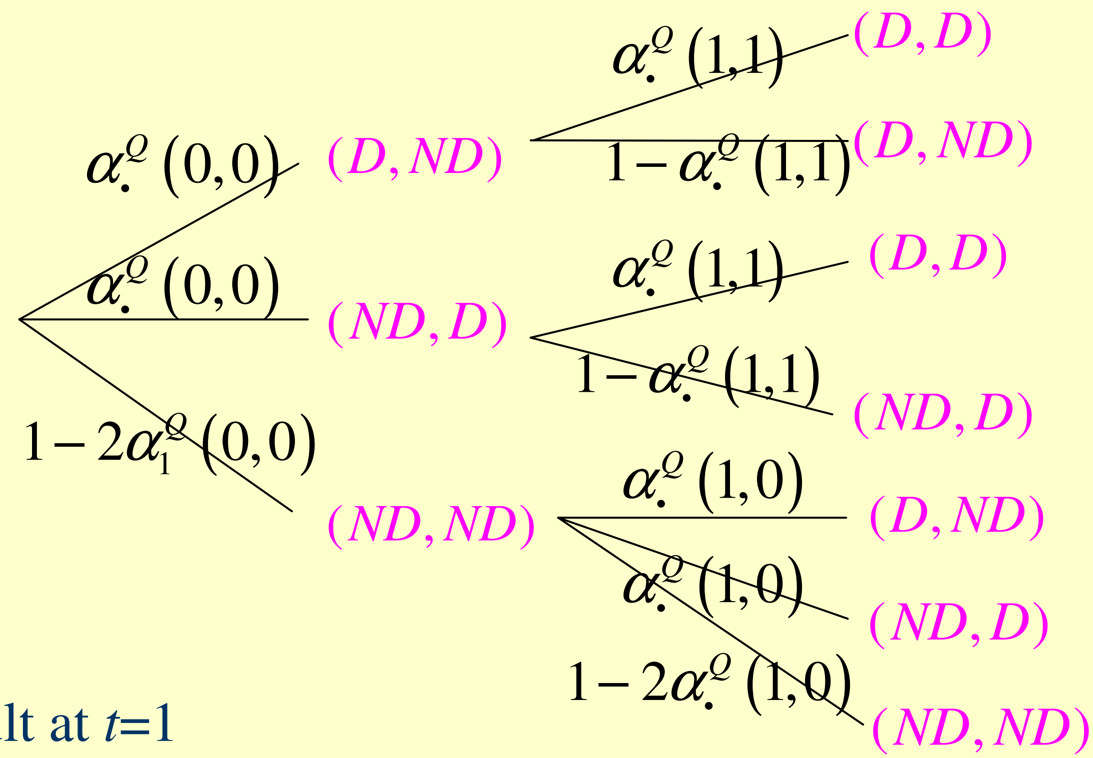
  - Possibility of taking into account discounting effects
  - The timing of premium payments
  - Computation of dynamic deltas with respect to short or actual CDS on names 1,2

Tree approach to hedging defaults

- In theory, one could also derive dynamic hedging strategies for index CDO tranches
  - Numerical issues: large dimensional, non recombining trees
  - Homogeneous Markovian assumption is very convenient
    - CDS premiums at a given time *t* only depend upon the current number of defaults $N(t)$
  - CDS premium at time 0 (no defaults) $\alpha_1^Q dt = \alpha_2^Q dt = \alpha_\bullet^Q \left(t = 0, N(0) = 0\right)$
  - CDS premium at time 1 (one default) $\lambda_1^Q dt = \lambda_2^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 1\right)$
  - CDS premium at time 1 (no defaults) $\pi_1^Q dt = \pi_2^Q dt = \alpha_\bullet^Q \left(t = 1, N(t) = 0\right)$

Tree approach to hedging defaults

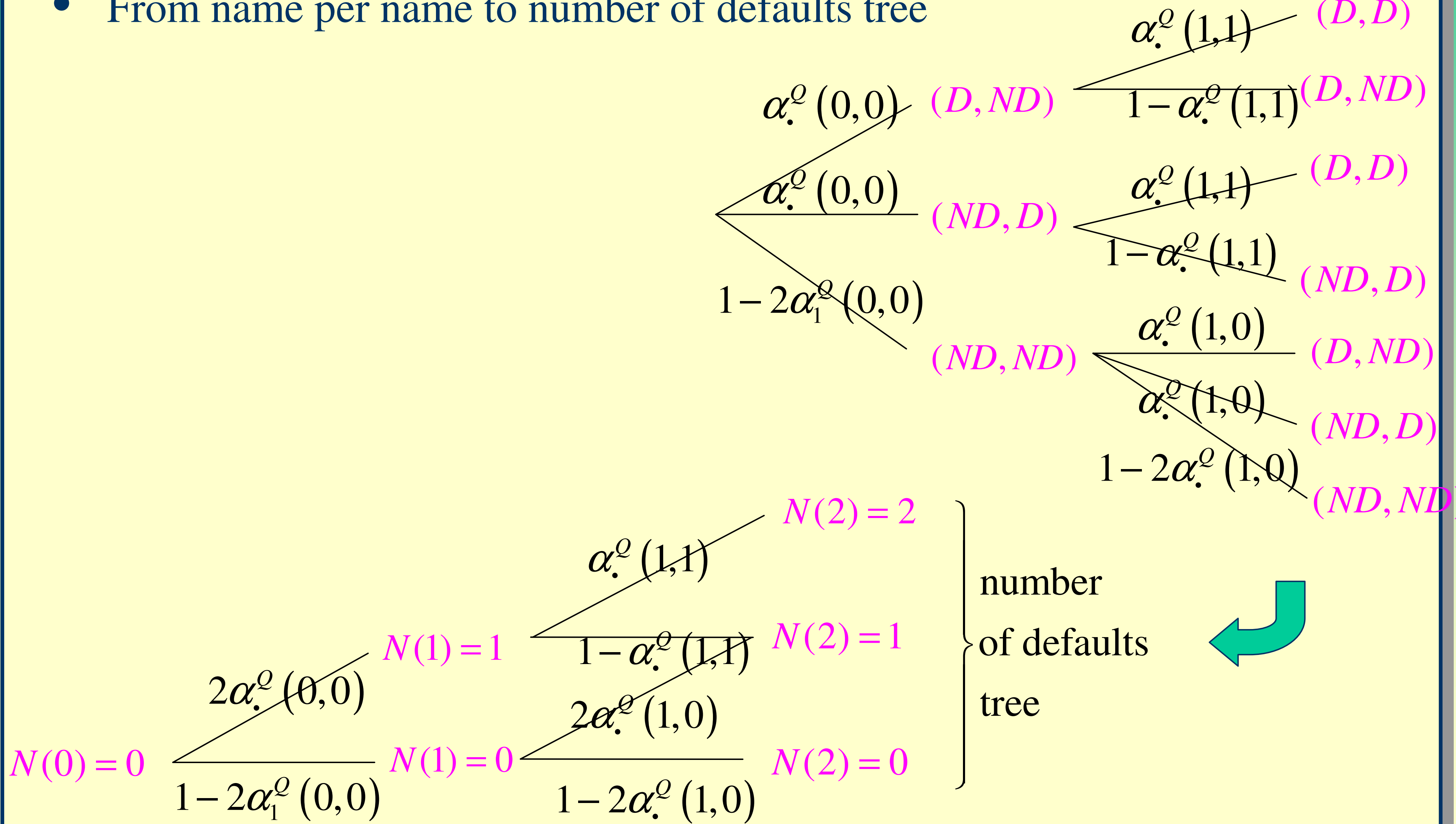
- Homogeneous Markovian tree



  - If we have $N(1)=1$, one default at t=1
  - The probability to have $N(2)=1$, one default at t=2…
  - Is $1-\alpha_{\cdot}^{Q}(1,1)$ and does not depend on the defaulted name at t=1
  - $N(t)$ is a Markov process
  - Dynamics of the number of defaults can be expressed through a binomial tree

Tree approach to hedging defaults

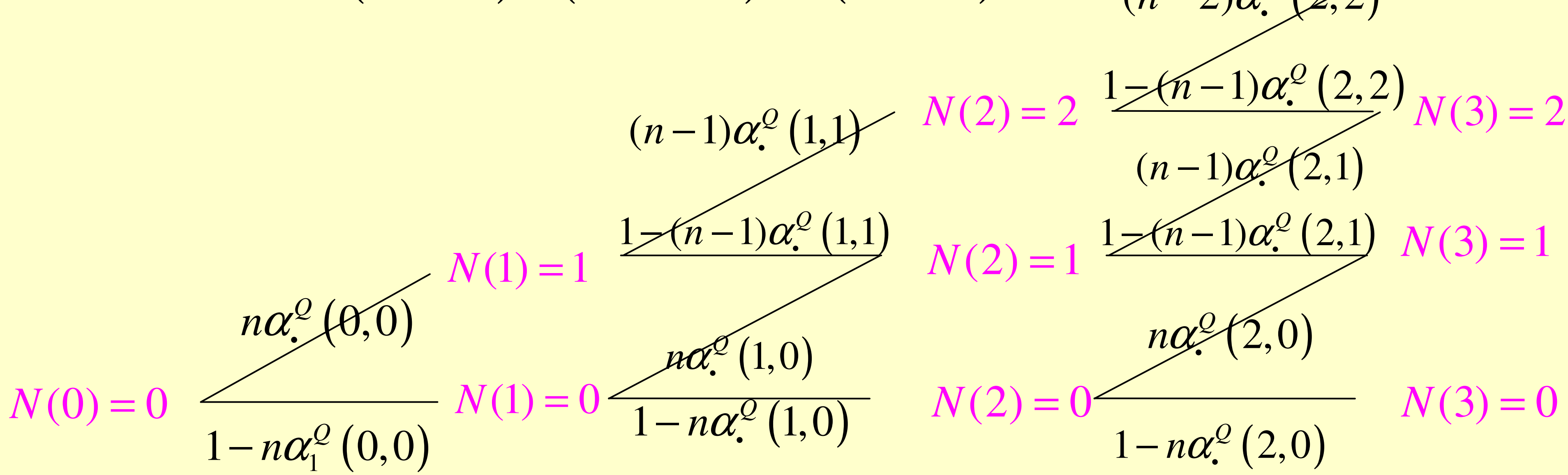
- From name per name to number of defaults tree



Tree approach to hedging defaults

- Easy extension to *n* names
  - Predefault name intensity at time *t* for $N(t)$ defaults: $\alpha_{\cdot}^{Q}\left(t, N(t)\right)$
  - Number of defaults intensity : sum of surviving name intensities:

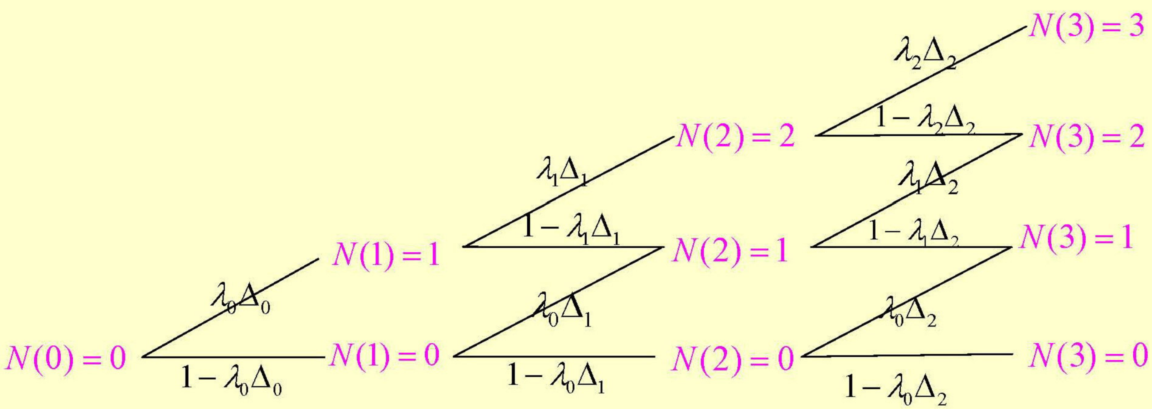
$$\lambda\left(t, N(t)\right) = \left(n - N(t)\right)\alpha_{\cdot}^{Q}\left(t, N(t)\right)$$

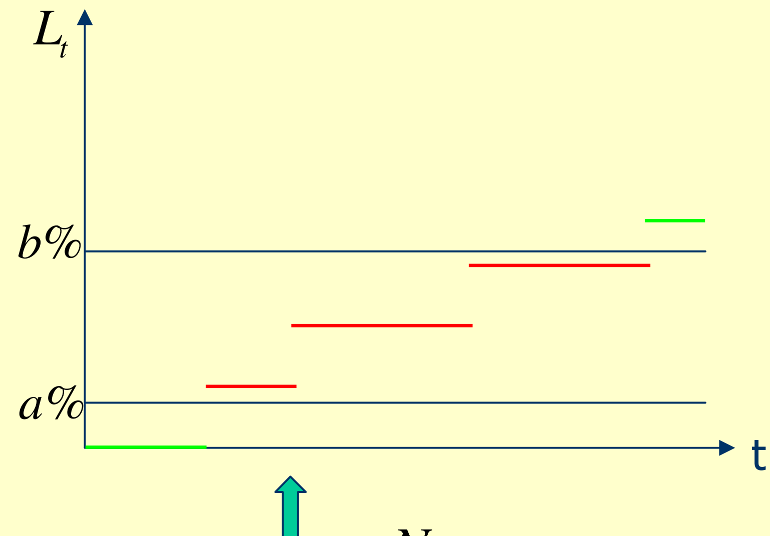


  - $\alpha_{\cdot}^{Q}(0,0), \alpha_{\cdot}^{Q}(1,0), \alpha_{\cdot}^{Q}(1,1), \alpha_{\cdot}^{Q}(2,0), \alpha_{\cdot}^{Q}(2,1), \ldots$ can be easily calibrated
  - on marginal distributions of $N(t)$ by forward induction.

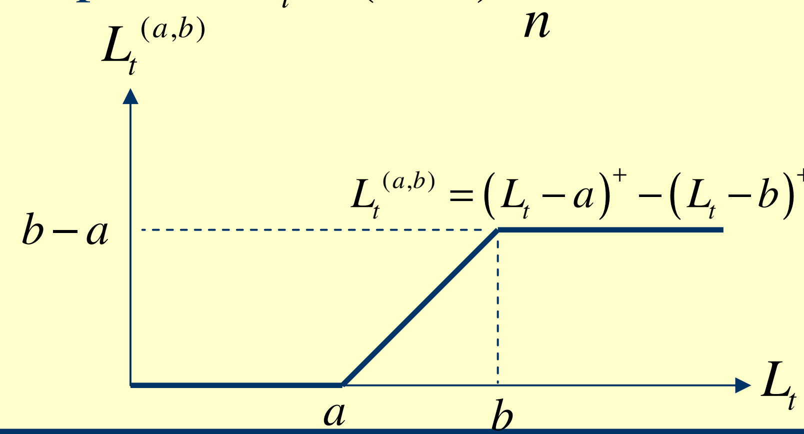
Tree approach to hedging defaults

- Previous recombining binomial risk-neutral tree provides a framework for the valuation of payoffs depending upon the number of defaults





- CDS Index : homogeneous portfolio of CDS
  - Cash-flows contingent to the aggregate loss process $L_t = (1-\delta)\frac{N_t}{n}$
  - where δ is the recovery rate
- CDO tranche [a%,b%]
  - Cash-flows contingent to $L_t^{(a,b)}$
  - Call spread option on the aggragate loss



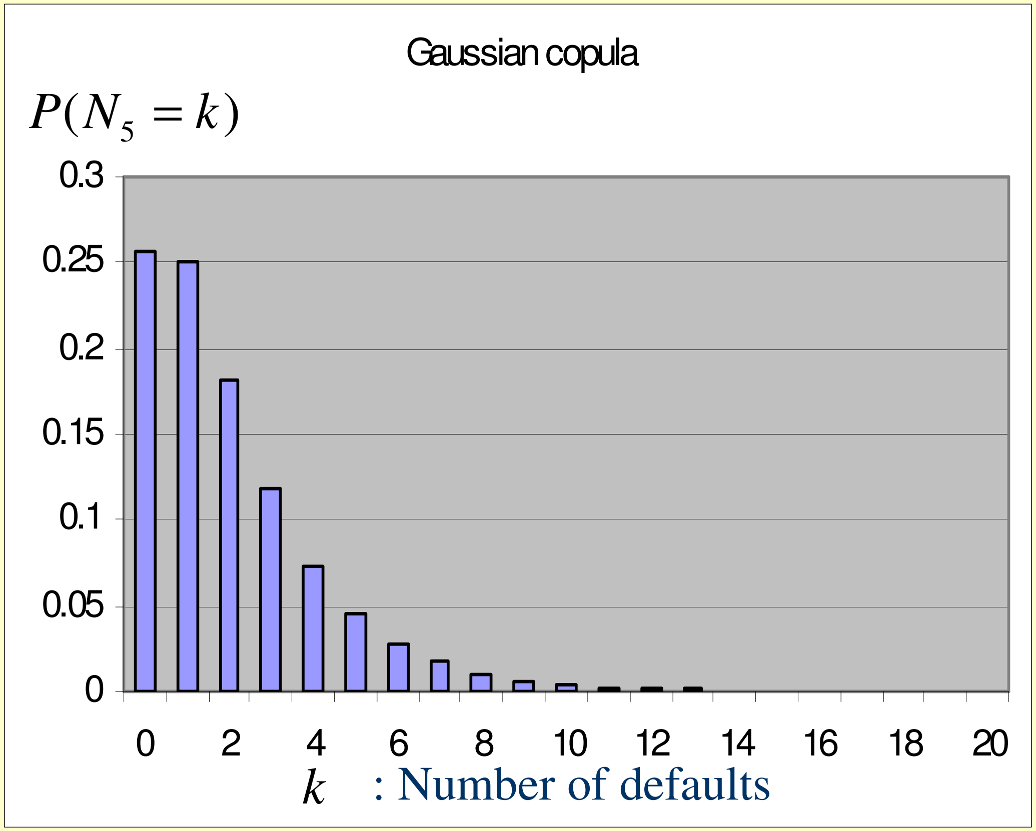
Results and comments

- What about the credit deltas?
  - In a homogeneous framework, deltas with respect to CDS are all the same
  - Perfect dynamic replication of a CDO tranche with a credit default swap index and the default-free asset
  - Credit delta with respect to the credit default swap index
  - = change in PV of the tranche / change in PV of the CDS index

Results and comments

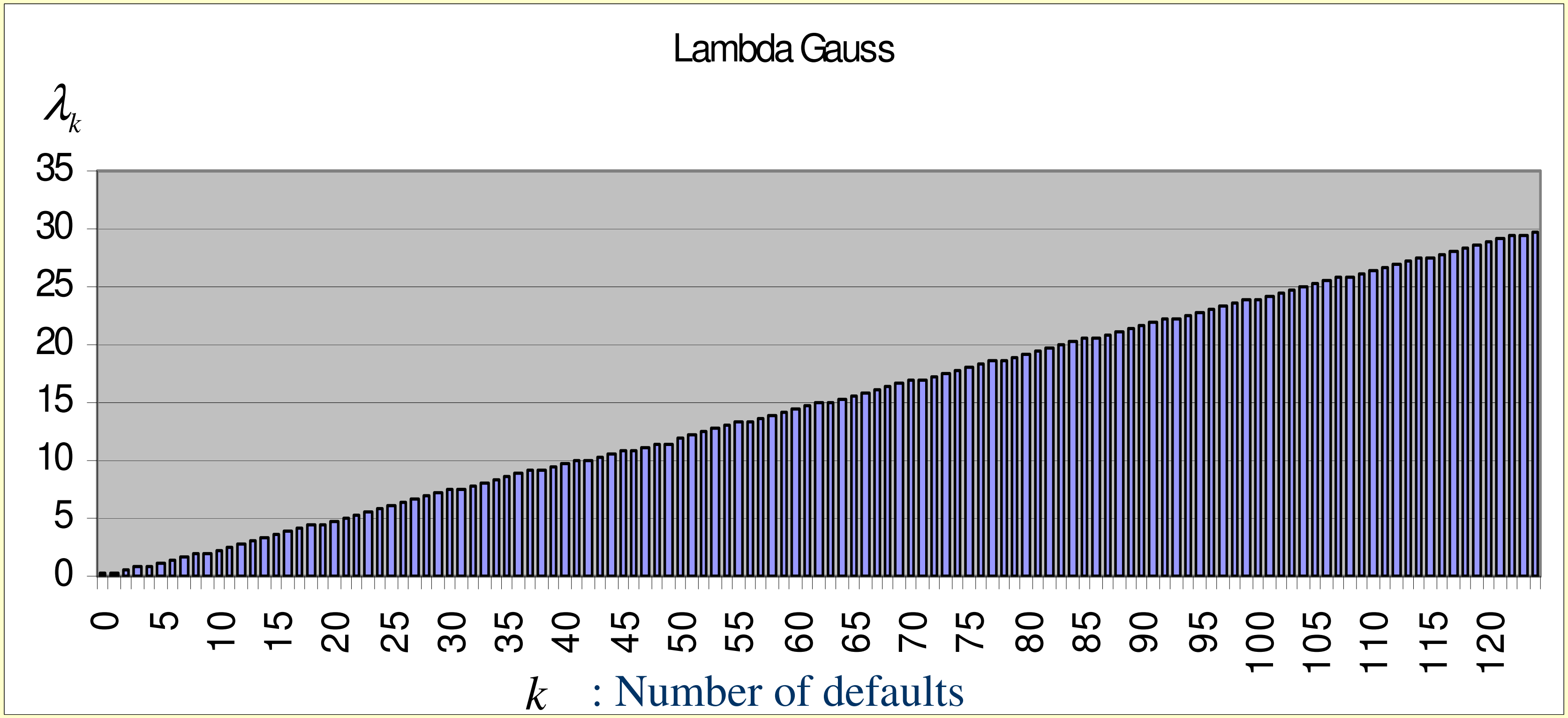
- Input: number of defaults distribution at 5Y generated from a Gaussian copula

  – Number of names: 125
  – Correlation parameter: 30%
  – Default-free rate: 3%
  – CDS spreads: 20bps per annum
    - **5Y default probability: 1.65%**
  – Recovery rate: 40%



Results and comments

- Calibration of loss intensities
  - For simplicity, assumption of time homogeneous intensities
  - Figure below represents loss intensities, with respect to the number of defaults
  - Increase in intensities: contagion effects



Results and comments

- Dynamics of the 5Y CDS index spread
  - In bp pa

		Weeks						
		0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	20	19	19	18	18	17	17
	1	0	31	30	29	28	27	26
	2	0	46	44	43	41	40	38
	3	0	63	61	58	56	54	52
	4	0	83	79	76	73	70	67
	5	0	104	99	95	91	87	83
	6	0	127	121	116	111	106	101
	7	0	151	144	138	132	126	120
	8	0	176	169	161	154	146	140
	9	0	203	194	185	176	168	160
	10	0	230	219	209	200	190	181
	11	0	257	246	235	224	213	203
	12	0	284	272	260	248	237	225
	13	0	310	298	286	273	260	248
	14	0	336	324	311	298	284	271
	15	0	0	348	336	323	308	294

Results and comments

- Dynamics of credit deltas:
  - **[0,3%] equity tranche**
  - With respect to the 5Y CDS index
  - For selected time steps

		OutStanding Nominal	Weeks						
			0	**14**	**28**	**42**	**56**	**70**	**84**
Nb Defaults	**0**	3.00%	0.814	0.843	0.869	0.893	0.915	0.933	0.949
	1	2.52%	0	0.614	0.658	0.702	0.746	0.787	0.827
	2	2.04%	0	0.341	0.384	0.431	0.482	0.535	0.591
	3	1.56%	0	0.140	0.165	0.194	0.229	0.269	0.315
	4	1.08%	0	0.045	0.054	0.064	0.078	0.095	0.117
	5	0.60%	0	0.013	0.015	0.017	0.020	0.024	0.030
	6	0.12%	0	0.002	0.002	0.002	0.003	0.003	0.003
	7	0.00%	0	0	0	0	0	0	0

  - Hedging strategy leads to a perfect replication of equity tranche payoff
  - When the number of defaults is > 6, the tranche is exhausted

Results and comments

		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
Nb Defaults	0	3.00%	0.814	0.843	0.869	0.893	0.915	0.933	0.949
	1	2.52%	0	0.614	0.658	0.702	0.746	0.787	0.827
	2	2.04%	0	0.341	0.384	0.431	0.482	0.535	0.591
	3	1.56%	0	0.140	0.165	0.194	0.229	0.269	0.315
	4	1.08%	0	0.045	0.054	0.064	0.078	0.095	0.117
	5	0.60%	0	0.013	0.015	0.017	0.020	0.024	0.030
	6	0.12%	0	0.002	0.002	0.002	0.003	0.003	0.003
	7	0.00%	0	0	0	0	0	0	0

- **Deltas are actually between 0 and 1**
- **Gradually decrease with the number of defaults**
  - Concave payoff, negative gammas
- **Credit deltas increase with time**
  - Consistent with a decrease in time value
  - At maturity date, when number of defaults < 6, delta=1

Results and comments

- Dynamics of credit deltas
  - **Junior mezzanine tranche [3,6%]**
  - Deltas lie in between 0 and 1
  - When the number of defaults is above 12, the tranche is exhausted

		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
Nb Defaults	0	3.00%	0.162	0.139	0.117	0.096	0.077	0.059	0.045
	1	3.00%	0	0.327	0.298	0.266	0.232	0.197	0.162
	2	3.00%	0	0.497	0.489	0.473	0.448	0.415	0.376
	3	3.00%	0	0.521	0.552	0.576	0.591	0.595	0.586
	4	3.00%	0	0.400	0.454	0.508	0.562	0.611	0.652
	5	3.00%	0	0.239	0.288	0.343	0.405	0.473	0.544
	6	3.00%	0	0.123	0.153	0.190	0.236	0.291	0.358
	7	2.64%	0	0.059	0.073	0.090	0.115	0.147	0.189
	8	2.16%	0	0.031	0.036	0.043	0.052	0.066	0.086
	9	1.68%	0	0.019	0.020	0.023	0.026	0.030	0.037
	10	1.20%	0	0.012	0.012	0.013	0.014	0.016	0.018
	11	0.72%	0	0.007	0.007	0.007	0.007	0.008	0.009
	12	0.24%	0	0.002	0.002	0.002	0.002	0.002	0.003
	13	0.00%	0	0	0	0	0	0	0

Results and comments

- **Dynamics of credit deltas (junior mezzanine tranche)**
  - Gradually increase and then decrease with the number of defaults
  - Call spread payoff (convex, then concave)
  - Initial delta = 16% (out of the money option)

		OutStanding Nominal	Weeks						
			0	14	28	42	56	70	84
Nb Defaults	0	3.00%	0.162	0.139	0.117	0.096	0.077	0.059	0.045
	1	3.00%	0	0.327	0.298	0.266	0.232	0.197	0.162
	2	3.00%	0	0.497	0.489	0.473	0.448	0.415	0.376
	3	3.00%	0	0.521	0.552	0.576	0.591	0.595	0.586
	4	3.00%	0	0.400	0.454	0.508	0.562	0.611	0.652
	5	3.00%	0	0.239	0.288	0.343	0.405	0.473	0.544
	6	3.00%	0	0.123	0.153	0.190	0.236	0.291	0.358
	7	2.64%	0	0.059	0.073	0.090	0.115	0.147	0.189
	8	2.16%	0	0.031	0.036	0.043	0.052	0.066	0.086
	9	1.68%	0	0.019	0.020	0.023	0.026	0.030	0.037
	10	1.20%	0	0.012	0.012	0.013	0.014	0.016	0.018
	11	0.72%	0	0.007	0.007	0.007	0.007	0.008	0.009
	12	0.24%	0	0.002	0.002	0.002	0.002	0.002	0.003
	13	0.00%	0	0	0	0	0	0	0

Results and comments

- Dynamics of credit deltas ([6,9%] tranche)
  - Initial credit deltas are smaller (further out of the money call spread)

Nb Defaults	OutStanding Nominal	Weeks						
		0	14	28	42	56	70	84
0	3.00%	0.017	0.012	0.008	0.005	0.003	0.002	0.001
1	3.00%	0	0.048	0.036	0.025	0.017	0.011	0.006
2	3.00%	0	0.133	0.107	0.083	0.061	0.043	0.029
3	3.00%	0	0.259	0.227	0.193	0.157	0.122	0.090
4	3.00%	0	0.371	0.356	0.330	0.295	0.253	0.206
5	3.00%	0	0.405	0.423	0.428	0.420	0.396	0.358
6	3.00%	0	0.346	0.392	0.433	0.465	0.482	0.481
7	3.00%	0	0.239	0.292	0.350	0.409	0.465	0.510
8	3.00%	0	0.139	0.181	0.232	0.293	0.363	0.436
9	3.00%	0	0.074	0.098	0.132	0.177	0.235	0.307
10	3.00%	0	0.042	0.053	0.070	0.095	0.132	0.183
11	3.00%	0	0.029	0.033	0.040	0.051	0.070	0.098
12	3.00%	0	0.025	0.026	0.028	0.033	0.040	0.053
13	2.76%	0	0.022	0.022	0.022	0.024	0.026	0.031
14	2.28%	0	0.020	0.018	0.018	0.018	0.019	0.020
15	1.80%	0	0	0.015	0.014	0.014	0.014	0.014
16	1.32%	0	0	0.013	0.011	0.010	0.010	0.010
17	0.84%	0	0	0.009	0.008	0.007	0.006	0.006
18	0.36%	0	0	0.005	0.004	0.003	0.003	0.003
19	0.00%	0	0	0	0	0	0	0

Conclusion

- What do we learn from this hedging approach?
  - Thanks to stringent assumptions:
    - **credit spreads driven by defaults**
    - **homogeneity**
    - **Markov property**
  - It is possible to compute a dynamic hedging strategy
    - **Based on the CDS index**
  - That fully replicates the CDO tranche payoffs
    - **Very simple implementation**
    - **Credit deltas are easy to understand**
  - Credit spread dynamics needs to be improved