Comparison results for exchangeable credit risk portfolios

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Contents

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   • De Finetti theorem and factor representation
   • Stochastic orders
   • Main results

2 Application to several popular CDO pricing models
   • Factor copula approaches
   • Structural model
   • Multivariate Poisson model
Homogeneity assumption: default indicators $D_1, \ldots, D_n$ form an exchangeable Bernoulli random vector

**Definition (Exchangeability)**

A random vector $(D_1, \ldots, D_n)$ is exchangeable if its distribution function is invariant for every permutations of its coordinates: $\forall \sigma \in S_n$

$$(D_1, \ldots, D_n) \overset{d}{=} (D_{\sigma(1)}, \ldots, D_{\sigma(n)})$$

- Same marginals
De Finetti theorem and factor representation

- Assume that $D_1, \ldots, D_n, \ldots$ is an exchangeable sequence of Bernoulli random variables.

- Thanks to de Finetti’s theorem, there exists a unique random factor $\tilde{p}$ such that $D_1, \ldots, D_n$ are conditionally independent given $\tilde{p}$.

- Denote by $F_{\tilde{p}}$ the distribution function of $\tilde{p}$, then:

$$P(D_1 = d_1, \ldots, D_n = d_n) = \int_0^1 p^\sum d_i (1 - p)^{n - \sum d_i} F_{\tilde{p}}(dp)$$

- $\tilde{p}$ is characterized by:

$$\frac{1}{n} \sum_{i=1}^n D_i \overset{a.s.}{\longrightarrow} \tilde{p} \text{ as } n \to \infty$$

- $\tilde{p}$ is exactly the loss of the infinitely granular portfolio (Basel 2 terminology).
Stochastic orders

- The convex order compares the dispersion level of two random variables
- Convex order: $X \leq_{cx} Y$ if $E[f(X)] \leq E[f(Y)]$ for all convex functions $f$
- Stop-loss order: $X \leq_{sl} Y$ if $E[(X - K)^+] \leq E[(Y - K)^+]$ for all $K \in \mathbb{R}$
  - $X \leq_{sl} Y$ and $E[X] = E[Y] \iff X \leq_{cx} Y$
Supermodular order

- The supermodular order captures the dependence level among coordinates of a random vector
- \((X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)\) if \(E[f(X_1, \ldots, X_n)] \leq E[f(Y_1, \ldots, Y_n)]\) for all supermodular functions \(f\)

**Definition (Supermodular function)**

A function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) is **supermodular** if for all \(x \in \mathbb{R}^n\), \(1 \leq i < j \leq n\) and \(\varepsilon, \delta > 0\) holds

\[
f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_n) \\
\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\]

**Müller (1997)**

*Stop-loss order for portfolios of dependent risks*

\[
(D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*) \Rightarrow \sum_{i=1}^{n} M_i D_i \leq_{sl} \sum_{i=1}^{n} M_i D_i^*
\]
Main results

Let us compare two credit portfolios with aggregate loss $L_t = \sum_{i=1}^{n} M_i D_i$ and $L^*_t = \sum_{i=1}^{n} M_i D^*_i$.

Let $D_1, \ldots, D_n$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$.

Let $D^*_1, \ldots, D^*_n$ exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$.

Theorem

$\tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D^*_1, \ldots, D^*_n)$

In particular, if $\tilde{p} \leq_{cx} \tilde{p}^*$, then:

- $E[(L_t - a)^+] \leq E[(L^*_t - a)^+]$ for all $a > 0$.
- $\rho(L_t) \leq \rho(L^*_t)$ for all convex risk measures $\rho$. 

Areski COUSIN and Jean-Paul LAURENT
Comparison results for exchangeable credit risk portfolios
Main results

- Let $D_1, \ldots, D_n, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}$
- Let $D_1^*, \ldots, D_n^*, \ldots$ be exchangeable Bernoulli random variables associated with the mixing probability $\tilde{p}^*$

**Theorem (reverse implication)**

$$ (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*), \forall n \in \mathbb{N} \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^*. $$
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   - De Finetti theorem and factor representation
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2 Application to several popular CDO pricing models
   - Factor copula approaches
   - Structural model
   - Multivariate Poisson model
Analysis of the dependence structure in several popular CDO pricing models

An increase of the dependence parameter leads to:

- a decrease of $[0\%, b]$ equity tranche premiums (which guaranties the uniqueness of the market base correlation)
- an increase of $[a,100\%]$ senior tranche premiums
The dependence structure of default times is described by some latent variables \( V_1, \ldots, V_n \) such that:

- \( V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i, \ i = 1 \ldots n \)
- \( V, \bar{V}_i, \ i = 1 \ldots n \) independent
- \( \tau_i = G^{-1}(H_\rho(V_i)), \ i = 1 \ldots n \)
  - \( G \): distribution function of \( \tau_i \)
  - \( H_\rho \): distribution function of \( V_i \)
- \( D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n \) are conditionally independent given \( V \)
- \( \frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid V] = P(\tau_i \leq t \mid V) = \bar{p} \)
Theorem

For any fixed time horizon \( t \), denote by \( D_i = 1_{\{\tau_i \leq t\}} \), \( i = 1 \ldots n \) and \( D_i^* = 1_{\{\tau_i^* \leq t\}} \), \( i = 1 \ldots n \) the default indicators corresponding to (resp.) \( \rho \) and \( \rho^* \), then:

\[
\rho \leq \rho^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- This framework includes popular factor copula models:
  - One factor Gaussian copula - the industry standard for the pricing of CDO tranches
  - Double t: Hull and White(2004)
  - NIG, double NIG: Guegan and Houdain(2005), Kalemanova, Schmid and Werner(2007)
  - Double Variance Gamma: Moosbrucker(2006)
Archimedean copula


- \( V \) is a positive random variable with Laplace transform \( \varphi^{-1} \)
- \( U_1, \ldots, U_n \) are independent Uniform random variables independent of \( V \)
- \( V_i = \varphi^{-1} \left( -\frac{\ln U_i}{V} \right) \), \( i = 1 \ldots n \) (Marshall and Olkin (1988))
  - \( (V_1, \ldots, V_n) \) follows a \( \varphi \)-archimedean copula
  - \( P(V_1 \leq v_1, \ldots, V_n \leq v_n) = \varphi^{-1}(\varphi(v_1) + \ldots + \varphi(v_n)) \)
- \( \tau_i = G^{-1}(V_i) \)
  - \( G \): distribution function of \( \tau_i \)
- \( D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n \) independent knowing \( V \)
- \( \frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\longrightarrow} E[D_i | V] = P(\tau_i \leq t | V) \)
Conditional default probability: $\tilde{p} = \exp \{ -\varphi(G(t)V) \}$

<table>
<thead>
<tr>
<th>Copula</th>
<th>Generator $\varphi$</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>$t^{-\theta} - 1$</td>
<td>$\theta \geq 0$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$(-\ln(t))^{\theta}$</td>
<td>$\theta \geq 1$</td>
</tr>
<tr>
<td>Franck</td>
<td>$-\ln \left[ \frac{(1 - e^{-\theta t})}{(1 - e^{-\theta})} \right]$</td>
<td>$\theta \in \mathbb{R}^*$</td>
</tr>
</tbody>
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Theorem

$\theta \leq \theta^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$
Archimedean copula

- Clayton copula
- Mixture distributions are ordered with respect to the convex order
Hull, Predescu and White (2005)

- Consider \( n \) firms
- Let \( V_{i,t}, \ i = 1 \ldots n \) be their asset dynamics

\[
V_{i,t} = \rho V_t + \sqrt{1 - \rho^2} \tilde{V}_{i,t}, \ i = 1 \ldots n
\]

- \( V, \tilde{V}_i, i = 1 \ldots n \) are independent standard Wiener processes
- Default times as first passage times:

\[
\tau_i = \inf \{ t \in \mathbb{R}^+ | V_{i,t} \leq f(t) \}, \ i = 1 \ldots n, \ f : \mathbb{R} \to \mathbb{R} \text{ continuous}
\]

- \( D_i = 1_{\{\tau_i \leq \tau\}}, i = 1 \ldots n \) are conditionally independent given \( \sigma(V_t, t \in [0, T]) \)
Theorem

For any fixed time horizon $T$, denote by $D_i = 1\{\tau_i \leq T\}$, $i = 1 \ldots n$ and $D_i^* = 1\{\tau_i^* \leq T\}$, $i = 1 \ldots n$ the default indicators corresponding to (resp.) $\rho$ and $\rho^*$, then:

$$\rho \leq \rho^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)$$
Structural model

- \( \frac{1}{n} \sum_{i=1}^{n} D_i \overset{a.s.}{\longrightarrow} \tilde{\rho} \)
- \( \frac{1}{n} \sum_{i=1}^{n} D_i^* \overset{a.s.}{\longrightarrow} \tilde{\rho}^* \)
- Empirically, mixture probabilities are ordered with respect to the convex order: \( \tilde{\rho} \leq_{\text{cx}} \tilde{\rho}^* \)

Distributions of Conditionnal Default Probabilities

Portfolio size=10000
\( X_0=0 \)
Threshold=-2
\( t=1 \) year
\( \delta t=0.01 \)
\( P(\tau_i \leq t)=0.033 \)
Multivariate Poisson model


- $\tilde{N}_t^i$ Poisson with parameter $\tilde{\lambda}$: idiosyncratic risk
- $N_t^i$ Poisson with parameter $\lambda$: systematic risk
- $(B_j^i)_{i,j}$ Bernoulli random variable with parameter $p$
- All sources of risk are independent
- $N_t^i = \tilde{N}_t^i + \sum_{j=1}^{N_t^i} B_j^i$, $i = 1 \ldots n$
- $\tau_i = \inf \{t > 0 | N_t^i > 0\}$, $i = 1 \ldots n$
Multivariate Poisson model

- Dependence structure of \((\tau_1, \ldots, \tau_n)\) is the Marshall-Olkin copula
- \(\tau_i \sim \text{Exp}(\bar{\lambda} + p\lambda)\)
- \(D_i = 1\{\tau_i \leq t\}, \ i = 1 \ldots n\) are conditionally independent given \(N_t\)
- \(\frac{1}{n} \sum_{i=1}^{n} D_i \xrightarrow{a.s.} E[D_i \mid N_t] = P(\tau_i \leq t \mid N_t)\)
- Conditional default probability:
  \[
  \tilde{p} = 1 - (1 - p)^{N_t} \exp(-\lambda t)
  \]
Comparison results for exchangeable credit risk portfolios

Multivariate Poisson model

- Comparison of two multivariate Poisson models with parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$

- Supermodular order comparison requires equality of marginals:
  \[ \bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda^* \]

- 3 comparison directions:
  - $p = p^*$: $\bar{\lambda}$ v.s $\lambda$
  - $\lambda = \lambda^*$: $\bar{\lambda}$ v.s $p$
  - $\bar{\lambda} = \bar{\lambda}^*$: $\lambda$ v.s $p$
Theorem \((p = p^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p\lambda^*\), then:

\[
\lambda \leq \lambda^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - a)^+]\):
  - 30 names
  - \(M_i = 1, \ i = 1 \ldots n\)
- When \(\lambda\) increases, the aggregate loss increases with respect to stop-loss order
**Theorem** ($\lambda = \lambda^*$)

*Let parameter sets $(\bar{\lambda}, \lambda, p)$ and $(\bar{\lambda}^*, \lambda^*, p^*)$ be such that $\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda$, then:

$$p \leq p^*, \quad \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{\lambda} \preceq_{cx} \bar{\lambda}^* \Rightarrow (D_1, \ldots, D_n) \preceq_{sm} (D_1^*, \ldots, D_n^*)$$

- Convex order for mixture probabilities

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**Multivariate Poisson model**

- **Application to several popular CDO pricing models**
- **Factor copula approaches**
- **Structural model**
- **Conclusion**

*Comparison results for exchangeable credit risk portfolios*
**Theorem \((\lambda = \lambda^*)\)**

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(\bar{\lambda} + p\lambda = \bar{\lambda}^* + p^*\lambda\), then:

\[
p \leq p^*, \; \bar{\lambda} \geq \bar{\lambda}^* \Rightarrow \bar{p} \leq_{\text{cx}} p^* \Rightarrow (D_1, \ldots, D_n) \leq_{\text{sm}} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(E[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \; i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Theorem \((\bar{\lambda} = \bar{\lambda}^*)\)

Let parameter sets \((\bar{\lambda}, \lambda, p)\) and \((\bar{\lambda}^*, \lambda^*, p^*)\) be such that \(p\lambda = p^*\lambda^*\), then:

\[
p \leq p^*, \lambda \geq \lambda^* \Rightarrow \tilde{p} \leq_{cx} \tilde{p}^* \Rightarrow (D_1, \ldots, D_n) \leq_{sm} (D_1^*, \ldots, D_n^*)
\]

- Computation of \(\mathbb{E}[(L_t - K)^+]\):
  - 30 names
  - \(M_i = 1, \; i = 1 \ldots n\)
- When \(p\) increases, the aggregate loss increases with respect to stop-loss order
Conclusion

- When considering an exchangeable vector of default indicators, the conditional independence assumption is not restrictive thanks to de Finetti’s theorem.
- The mixing probability (the factor) can be viewed as the loss of an infinitely granular portfolio.
- We completely characterize the supermodular order between exchangeable default indicator vectors in terms of the convex ordering of corresponding mixing probabilities.
- We show that the mixing probability is the key input to study the impact of dependence on CDO tranche premiums.
- Comparison analysis can be performed with the same method within a large class of CDO pricing models.
Exchangeability: how realistic is a homogeneous assumption?

- Homogeneity of default marginals is an issue when considering the pricing and the hedging of CDO tranches
  - ex: Sudden surge of GMAC spreads in the CDX index in May, 2005
  - This event dramatically impacts the equity tranche compared to others tranches
- But composition of standard indices are updated every semester, resulting in an increase of portfolio homogeneity
- It may be reasonable to split a credit portfolio in several homogeneous sub-portfolios (by economic sectors for example)
  - Then, for each sub-portfolio, we can find a specific factor and apply the previous comparison analysis
  - The initial credit portfolio can thus be associated with a vector of factors (one by sector)
  - Is it possible to relate comparison between global aggregate losses to comparison between vectors of random factors?
Are comparisons in a static framework restrictive?

- Are comparisons among aggregate losses at fixed horizons too restrictive?
- Computation of CDO tranche premiums only requires marginal loss distributions at several horizons
  - Comparison among aggregate losses at different dates is sufficient
- However, comparison of more complex products such as options on tranche or forward started CDOs are not possible in this framework
- Building a framework in which one can compare directly aggregate loss processes is a subject of future research