Comparison analysis of two alternative hedging methods for CDO tranches

Areski COUSIN

Université d’Evry Val d’Essonne

Groupe de travail - Evry 12 March 2009
Joint work with Stéphane Crépey
Introduction and motivation

Recent financial turmoil has deeply affected the market of structured credit derivatives

CDS index products are still liquid but ...

Investors have more incentive to risk-manage their trading books

However, standard hedging methods have revealed serious drawbacks during the crisis

- focus on the computation of spread sensitivities (credit deltas)
- within a static Gaussian copula model
- does not rely on a sound theory of replication
- negative deltas may occur in a steep base correlation market:

  Morgan and Mortensen (2007)
Introduction and motivation

- We consider discrete-time hedging of index CDO tranches using
  - the underlying CDS index
  - the risk-free asset
- Hedging consists in taking complementary positions in the index and in the risk-free asset in order to minimize the overall evolution of market prices.
  - These positions need to be regularly updated over time
- **Aim of the presentation**: performance analysis of two alternative hedging strategies
  - $\Delta^{lo}$: delta of the tranche within a Markovian contagion model
  - $\Delta^{li}$: delta of the tranche within a Gaussian copula model
In the literature

- Crépey (2004) performs a similar analysis for the equity market
  - Comparison of hedging performance of equity options using two alternative deltas: Black-Scholes implied delta and local volatility delta
  - He exhibits two market directions: (slow/fast) and (rallies/sell-offs)
  - Negatively skew market: local volatility delta provides a better hedge than implied delta during slow rallies or fast sell-offs and a worse hedge during fast rallies and slow sell-offs.
- Analogies with the credit market are not so obvious
  - Interaction between default risk and spread risk, large dimension of the portfolio, recovery rate uncertainty
In the literature

- Laurent, Cousin and Fermanian (2007) study the hedging of index CDO tranche in a Markovian contagion model using the CDS index.
  - When simultaneous defaults are precluded, the CDO tranche market is complete.
  - Computation of dynamic hedging strategies along the nodes of a binomial tree.
  - $\Delta^{lo}$ seems to be smaller than $\Delta^{li}$ for equity tranches and higher for more senior tranches when the two models are calibrated to the same market data.
  - Do not study in details the performance of these two hedging strategies.
In the literature

- **Cont and Kan (2008)** perform an empirical comparison of various hedging strategies for index CDO tranches
  - three different notions of deltas: spread-delta, jump-to-default delta, quadratic risk-minimization delta
  - deltas computed in various models calibrated to the same set of market data
  - Backtest the strategies before and during the crisis

- Main conclusions:
  - spread-deltas are very similar across models calibrated to the same data set
  - jump-to-default ratios are significatively different across models (substantial model risk)
  - spread-deltas hedges are preferred before the crisis and not preferred during the crisis

- But study of performance only for a single (market) trajectory
- Do not address the issue of hedging with individual CDS
Slice the credit portfolio into different risk levels or **CDO tranches**

- ex: CDO tranche on **standardized Index** such as CDX North America or Itraxx Europe

### Diagram

- Credit risk portfolio
  - Ex: Itraxx 125 names
  - Losses:
    - Super Senior: 20%
    - Senior: 12%
    - Mezzanine: 9%
    - Junior Mezzanine: 6%
    - Equity: 3%

Notations

- Credit portfolio with $n$ reference entities
- $\tau_1, \ldots, \tau_n$: default times
- $R$: homogeneous and constant recovery rate at default ($R = 40\%$ typically)
- Number of defaults process:

$$N_t = \sum_{i=1}^{n} 1\{\tau_i \leq t\}$$

- CDO tranche cash-flows are driven by the aggregate loss process normalized to unity:

$$L_t = \frac{1}{n} (1 - R) N_t$$

- Cash-flows only depends on $\phi(L_t)$, $0 \leq t \leq T$
Assumptions

- By simplicity, we consider zero interest rate $r = 0$
- Stylized products with simplified cash-flows:
  - Cash-flows of the index and associated CDO tranches only depend on $\phi(L_T)$
  - Protection or default payment only occur at maturity $T$
  - The premium leg is paid upfront
- Given a risk-neutral probability $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$
  - The time $t$ price of a derivative with a $\mathcal{F}_T$-measurable and bounded payoff $\xi = \phi(L_T)$ is:
    $$\mathbb{E}[\xi \mid \mathcal{F}_t]$$
CDS Index and standardized CDO tranches

- Loss on CDO tranche \([a, b]\): \(L_T^{[a,b]}\) has a call spread payoff with respect to \(L_T\):

\[
L_T^{[a,b]}(L_T - b)^+ - (L_T - a)^+ = \phi^{[a,b]}(N_T)
\]

- Loss on CDO tranche \([a, b]\):

\[
L_T^{[a,b]} = (L_T - a)^+ - (L_T - b)^+ = \phi^{[a,b]}(N_T)
\]

- equity tranches: \(a = 0\%\) and \(L_t^{[0,b]} = \min(L_T, b)\)
- senior tranches: \(b = 100\%\) and \(L_t^{[a,1]} = (L_T - a)^+\)
- CDS index associated with a \([0\%, 100\%]\) CDO tranche
As the premium leg is paid upfront, the analysis is focused on the **protection leg**

The time-$t$ cum-dividend price of a stylized CDO tranche \([a, b]\) (protection leg) is referred to as:

\[
\Pi_t = \mathbb{E} \left[ \phi^{[a,b]}(N_T) \mid \mathcal{F}_t \right]
\]

The time-$t$ cum-dividend price of the stylized underlying index (protection leg) is referred to as:

\[
P_t = \mathbb{E} \left[ \phi^{[0,1]}(N_T) \mid \mathcal{F}_t \right]
\]
Homogeneous one factor Gaussian copula model

- Also referred to as the **Li model**

\[ V_i = \rho V + \sqrt{1 - \rho^2} \bar{V}_i, \ i = 1 \ldots n \]: latent variables
  - \( V, \bar{V}_i, i = 1 \ldots n \): independent Gaussian random variables

- Default times defined by:
  \[ \tau_i = F_i^{-1}(\Phi(V_i)), \ i = 1 \ldots n \]
  - \( F_1 = \ldots = F_n = F \): cdf of \( \tau_i, i = 1, \ldots, n \)
  - \( \Phi \): cdf of \( V_i \)

- Conditional default probability

\[ p_t(V) = \mathbb{P}(\tau_i \leq t \mid V) = \Phi \left( \frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1 - \rho^2}} \right) \]

- Loss distribution is merely a binomial mixture:

\[ \mathbb{P}(N_t = k) = \left( \begin{array}{c} n \\ k \end{array} \right) \int p_t(x)^k (1 - p_t(x))^{n-k} \nu(x) dx, \ k = 0, \ldots, n \]
At each time $t$, the model parameters $\rho_t$ and $F_t$ are calibrated on market spreads

$F_t$ is inferred from the term structure of index spreads at time $t$

- Index spread curve assumed to be flat and equal to $S_t$

$$F_t(s) = \mathbb{P}(\tau_i \leq t) = 1 - \exp \left( -\frac{S_t}{1 - R} (s - t) \right), \ s \geq t$$

One dependence parameter $\rho^b_t$ associated with each base tranche $[0, b]$, $b = 3\%, 6\%, 9\%, 12\%, 22\%$ (iTraxx)

- $\Pi_{t}^{ma}(T, a, b)$: market price of CDO tranche $[a, b]$, maturity $T$
- $\Pi^{li}(T, a, b; t, S_t, \rho_t)$: price of CDO tranche $[a, b]$ in the Li model
- Base correlation $\rho^b_t$ is such that:

$$\Pi^{li}(T, 0, b; t, S_t, \rho^b_t) = \Pi_{t}^{ma}(T, 0, b)$$
Homogeneous one factor Gaussian copula model

Monotonic base tranche and senior tranche prices with respect to $\rho$ in the Li model

$$\frac{\partial \Pi^{li}(T, 0, b; t, S_t, \rho)}{\partial \rho} \leq 0, \quad \frac{\partial \Pi^{li}(T, a, 1; t, S_t, \rho)}{\partial \rho} \geq 0, \quad \forall a, b \in [0, 1]$$

Given the existence of base correlations $\rho^b_t$, these parameters are unique.

CDO tranche market typically reflects steep base correlation curves:
Homogeneous Markovian contagion model

- Also referred to as the **local intensity model**
- Dynamic model where the cumulative default intensity only depends on number of defaults
- \(N_t\) is a continuous-time Markov chain with generator matrix:

\[
\Lambda(t) = \begin{pmatrix}
-\lambda(t,0) & \lambda(t,0) & 0 & 0 \\
0 & -\lambda(t,1) & \lambda(t,1) & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -\lambda(t,n-1) & \lambda(t,n-1)
\end{pmatrix}
\]

- \(\Pi^{lo}(t,T) = \mathbb{E}[\Phi(N_T) \mid \mathcal{F}_t] = \mathbb{E}[\Phi(N_T) \mid N_t]\)
Homogeneous Markovian contagion model

- Vector of prices $\Pi^{lo}(t, T) = (\Pi^{lo}_1(t, T), \ldots, \Pi^{lo}_n(t, T))^\top$
  - where $\Pi^{lo}_i(t, T) = \mathbb{E} [\Phi(N_T) \mid N_t = i], \ i = 1, \ldots, n$
- can be related to the vector of terminal payoffs $C = (\Phi(0), \ldots, \Phi(n))^\top$
- using the backward Kolmogorov equation:

$$\left\{\begin{array}{c}
\frac{\partial \Pi^{lo}(t, T)}{\partial t} = -\Lambda(t) \Pi^{lo}(t, T) \\
\Pi^{lo}(T, T) = C
\end{array}\right.$$ 

- When the intensities are time-homogeneous, i.e, $\Lambda(t) = \Lambda$ then:

$$\Pi^{lo}(t, T) = \exp \left((T - t)\Lambda\right) C$$

An investor enters a sell-protection position on a CDO tranche \([a, b]\)

He wants to cover his position until an hedging horizon: \(T_1 \leq T\)

**Delta-hedging** the tranche consists in rebalancing a complementary position in a portfolio including the underlying index and the risk-free asset

- at every point in time of a subdivision \(0 = t_0 \leq t_1 \leq \cdots \leq t_p = T_1\)
- index position determined in order to minimize the overall exposure
Delta-hedging in discrete time

- The profit-and-loss (P&L) trajectory $e = (e_{t_k})_{0 \leq k \leq p}$ is obtained by adding up the P&L increments:
  $$\delta_k e = -\delta_k \Pi + \Delta_{t_k} \delta_k P$$

- $\delta_k \Pi = \Pi_{t_{k+1}} - \Pi_{t_k}$: increments of the tranche market price in $(t_k, t_{k+1}]$
- $\delta_k P = P_{t_{k+1}} - P_{t_k}$: increments of the index market price in $(t_k, t_{k+1}]$
- $\Delta_{t_k}$: number of units of index contract in the hedging portfolio over the time interval $(t_k, t_{k+1}]$

- Aim is to compare the P&L trajectory $e$ obtained using two strategies:
  - $\Delta = \Delta^{lo}$: delta of the tranche in a Markovian contagion model
  - $\Delta = \Delta^{li}$: delta of the tranche in a Gaussian copula model
Delta-hedging in discrete time

- $\Delta^{lo}$: jump-to-default in the local intensity model

$$\Delta_{t}^{lo} = \frac{\Pi_{i+1}^{lo}(t) - \Pi_{i}^{lo}(t)}{P_{i+1}^{lo}(t) - P_{i}^{lo}(t)}$$

- where $\Pi_{i}^{lo}(t) = \mathbb{E} \left[ \Phi^{[a,b]}(N_{T}) \mid N_{t} = i \right]$ 
- and $P_{i}^{lo}(t) = \mathbb{E} \left[ \Phi^{[0,1]}(N_{T}) \mid N_{t} = i \right]$

- $\Delta^{li}$: spread-delta (sticky strike rule) in the Li model

$$\Delta_{t}^{li} = \frac{\Pi^{li}(t, S_{t} + \varepsilon, \rho_{t}) - \Pi^{li}(t, S_{t}, \rho_{t})}{P^{li}(t, S_{t} + \varepsilon) - P^{li}(t, S_{t})}$$

- $\varepsilon$ is typically equal to some few basis points
- The two models are calibrated on the same set of market spreads at every time $t_{k}$, $k = 0, \ldots, p - 1$
Methodology similar to Hull and Suo (2000) and Crépey (2004)

Theoretical market given as a fixed Markovian contagion model:

- Market prices are such that:
  \[
  \Pi_t = \Pi^{lo}(t, N_t), \quad P_t = P^{lo}(t, N_t), \quad S_t = S^{lo}(t, N_t), \quad \rho_t = \rho^{lo}(t, N_t)
  \]

- Given \( N_t = i \), \( \Pi_t = \Pi^{lo}_i(t) \), \( P_t = P^{lo}_i(t) \), \( S_t = S^{lo}_i(t) \), \( \rho_t = \rho^{lo}_i(t) \)

As the CDO tranche market is complete in homogeneous Markovian contagion model, \( \Delta^{lo} \) is the perfect continuous-time hedging strategy

But we consider hedging in discrete time . . .
Analysis in a Markovian contagion model

• Simulation of \( \bar{N} \) default trajectories in the local intensity model
  • Simulation of \( (\tau^{(1)}, \ldots, \tau^{(n)}) \), \( j = 1, \ldots, \bar{N} \)

• Without loss of generality, we focus the hedging analysis on a single period \( (t_k, t_{k+1}] \)

• \( \Delta_{t_k}^{lo} \) is preferred to \( \Delta_{t_k}^{li} \) on the period \( (t_k, t_{k+1}] \) if

\[
\text{Var}(\delta_k e^{lo}) < \text{Var}(\delta_k e^{li})
\]

where \( \delta_k e^{lo} \) is the P&L increment in \( (t_k, t_{k+1}] \) using \( \Delta_{t_k}^{lo} \)

where \( \delta_k e^{li} \) is the P&L increment in \( (t_k, t_{k+1}] \) using \( \Delta_{t_k}^{li} \)
As in Crépey (2004) we distinguish two market directions (slow/fast) and (rallies/sell-offs).

Regarding the period \( (t_k, t_{k+1}] \), a market trajectory is said to be:

- **Fast**: a default is observed on the period \( (t_k, t_{k+1}] \)
- **Slow**: no default is observed on the period \( (t_k, t_{k+1}] \)
- **Rallies**: \( \delta_k P \leq 0 \) (decreasing index spread)
- **Sell-offs**: \( \delta_k P \geq 0 \) (increasing index spread)
We consider the hedging of an **equity tranche** (Analysis is similar for a senior tranche)

**Proposition**

\( \delta_k e^{lo} \) is **positive** at slow market regimes and **negative** at fast market regimes

Indeed, one can remark that:

\[
\delta_k e^{lo} = -\delta_k \Pi + \Delta^{lo}_{t_k} \delta_k P = \int_{t_k}^{t_k+1} \left( \Delta^{lo}_{t_k} - \Delta^{lo}_{t} \right) dP_t
\]

- Consider a small interval \((t_k, t_k+1]\)
- no default (slow): \(dP_t \simeq \delta_k P \leq 0\) (time decay effect)
- one default (fast): \(dP_t \simeq \delta_k P \geq 0\) (cash-flow and contagion effect)
And $\Delta_{t}^{lo}$ is typically increasing in $t$:

$$(t, i) \rightarrow \Delta^{lo}(t, i) = \frac{\Pi_{i+1}^{lo}(t) - \Pi_{i}^{lo}(t)}{P_{i+1}^{lo}(t) - P_{i}^{lo}(t)}, \quad 0 \leq t \leq 5, \quad i = 0, \ldots, 6$$
Analysis in a Markovian contagion model

- **Ordering of the two deltas** for equity tranche

**Proposition**

- If \( \rho_{i+1}^{lo}(t) \geq \rho_i^{lo}(t) \), then one may expect that \( \Delta_t^{lo} \leq \Delta_t^{li} \)
- If \( \rho_{i+1}^{lo}(t) \leq \rho_i^{lo}(t) \), then one may expect that \( \Delta_t^{lo} \geq \Delta_t^{li} \)

Indeed, by definition of the implied base correlation:

\[
\Pi_{i+1}^{lo}(t) - \Pi_i^{lo}(t) = \Pi^{li}(t, S_{i+1}^{lo}(t), \rho_{i+1}^{lo}(t)) - \Pi^{li}(t, S_i^{lo}(t), \rho_i^{lo}(t)) \\
= \Pi^{li}(t, S_{i+1}^{lo}(t), \rho_{i+1}^{lo}(t)) - \Pi^{li}(t, S_{i+1}^{lo}(t), \rho_i^{lo}(t)) \\
+ \Pi^{li}(t, S_i^{lo}(t), \rho_i^{lo}(t)) - \Pi^{li}(t, S_i^{lo}(t), \rho_i^{lo}(t))
\]

But as \( \partial_\rho \Pi^{li}(t, S, \rho) \leq 0 \) for an equity tranche:

\[
\Pi^{li}(t, S_{i+1}^{lo}(t), \rho_{i+1}^{lo}(t)) \leq \Pi^{li}(t, S_i^{lo}(t), \rho_i^{lo}(t))
\]
Ordering of the two deltas (cont.)

By definition of the local intensity delta:

\[
\Delta^{lo}(t,i) = \frac{\Pi^{lo}_{i+1}(t) - \Pi^{lo}_i(t)}{P^{lo}_{i+1}(t) - P^{lo}_i(t)}
\]

\[
\leq \frac{\Pi^{li}(t, S^{lo}_{i+1}(t), \rho^{lo}_i(t)) - \Pi^{li}(t, S^{lo}_i(t), \rho^{lo}_i(t))}{P^{lo}_{i+1}(t) - P^{lo}_i(t)}
\]

\[
= \frac{\Pi^{li}(t, S^{lo}_{i+1}(t), \rho^{lo}_i(t)) - \Pi^{li}(t, S^{lo}_i(t), \rho^{lo}_i(t))}{P^{li}(t, S^{lo}_{i+1}(t)) - P^{li}(t, S^{lo}_i(t))}
\]

\[
\leq \Delta^{li}_t
\]
Analysis in a Markovian contagion model

- **Comparison of P&L increments** obtained using $\Delta^{lo}$ and $\Delta^{li}$

  $$\delta e^{li} = \delta e^{lo} + \left(\Delta^{li} - \Delta^{lo}\right) \delta P$$

- In a market where $\rho^{lo}_{i+1}(t) \geq \rho^{lo}_{i}(t)$ (steep base correlation market)

  - [0% – b] equity tranche:

    | Market regime | Rally | Sell-Off |
    |----------------|-------|----------|
    | Slow           | $(\delta e^{li})^+ \leq \delta e^{lo}$ |          |
    | Fast           |       | $\delta e^{lo} \leq - (\delta e^{li})^-$ |

  - [a – 100%] senior tranche:

    | Market regime | Rally | Sell-Off |
    |----------------|-------|----------|
    | Slow           | $\delta e^{lo} \leq - (\delta e^{li})^-$ |          |
    | Fast           |       | $(\delta e^{li})^+ \leq \delta e^{lo}$ |
Analysis in a Markovian contagion model

- **Comparison of P&L increments** obtained using $\Delta^{lo}$ and $\Delta^{li}$

\[
\delta e^{li} = \delta e^{lo} + (\Delta^{li} - \Delta^{lo}) \delta P
\]

- In a market where $\rho^{lo}_{i+1}(t) \leq \rho^{lo}_i(t)$ (flat base correlation market)
- [0% – b] equity tranche:

<table>
<thead>
<tr>
<th>Market regime</th>
<th>Rally</th>
<th>Sell-Off</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>$0 \leq \delta e^{lo} \leq \delta e^{li}$</td>
<td></td>
</tr>
<tr>
<td>Fast</td>
<td></td>
<td>$\delta e^{li} \leq \delta e^{lo} \leq 0$</td>
</tr>
</tbody>
</table>

- [a – 100%] senior tranche:

<table>
<thead>
<tr>
<th>Market regime</th>
<th>Rally</th>
<th>Sell-Off</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>$\delta e^{li} \leq \delta e^{lo} \leq 0$</td>
<td></td>
</tr>
<tr>
<td>Fast</td>
<td></td>
<td>$0 \leq \delta e^{lo} \leq \delta e^{li}$</td>
</tr>
</tbody>
</table>

- $\Delta^{lo}$ provides a better hedge than $\Delta^{li}$
Numerical results

- Sell-protection position in a [0-3%] equity tranche
- We numerically compare hedging performance of $\Delta^{li}$ and $\Delta^{lo}$ in two different markets:
  - One with a high increase of contagion (red)
  - One with a low increase of contagion (blue)
The high contagion market features an increase of base correlation at the arrival of defaults:

$$\rho_{i+1}^{lo}(t) \geq \rho_{i}^{lo}(t)$$

![Graph showing numerical results for iTraxx detachment points with different numbers of defaults.](image-url)
Numerical results (high contagion market)

- **Histogram of P&L increments** using $\Delta^{lo}$ (left) and $\Delta^{li}$ (right)

- Hedging period: $[0, 0.02]$ (one week)
- $\tilde{N} = 10000$ trajectories
Numerical results (high contagion market)

- **Histogram of P&L increments** using $\Delta_{lo}$ (left) and $\Delta_{li}$ (right)
- Hedging period: $[0, 0.09]$ (one month)
- $\bar{N} = 10000$ trajectories
Numerical results (high contagion market)

- **Histogram of P&L increments using** $\Delta^{lo}$ (left) and $\Delta^{li}$ (right)
- **Hedging period**: $[0, 0.5]$ (one semester)
- $\bar{N} = 10000$ trajectories

![Histogram of P&L increments](image)
Numerical results (high contagion market)

- Standard deviation of P&L increments function of the hedging horizon
  - $\tilde{N} = 10000$ simulations at each time step
  - $\sigma(\delta P&L)$ using $\Delta^{lo}$ in blue
  - $\sigma(\delta P&L)$ using $\Delta^{li}$ in red
The low contagion market features a decrease of base correlation at the arrival of defaults:

$$\rho_{i+1}^{lo}(t) \leq \rho_{i}^{lo}(t)$$
Numerical results (low contagion market)

- **Histogram of P&L increments** using $\Delta^{lo}$ (left) and $\Delta^{li}$ (right)
- **Hedging period**: $[0, 0.02]$ (one week)
- $\tilde{N} = 10000$ trajectories
Numerical results (low contagion market)

- **Histogram of P&L increments** using $\Delta^{lo}$ (left) and $\Delta^{li}$ (right)
  - Hedging period: $[0, 0.09]$ (one month)
  - $\bar{N} = 10000$ trajectories
**Numerical results (low contagion market)**

- **Histogram of P&L increments** using $\Delta^{lo}$ (left) and $\Delta^{li}$ (right)
  - Hedging period: $[0, 0.5]$ (one semester)
  - $\bar{N} = 10000$ trajectories

![Histogram of P&L increments](attachment:image.png)
Numerical results (low contagion market)

- Standard deviation of P&L increments function of the hedging horizon
  - $\bar{N} = 10000$ simulations at each time step
  - $\sigma(\delta P&L)$ using $\Delta^{lo}$ in blue
  - $\sigma(\delta P&L)$ using $\Delta^{li}$ in red

![Graph showing the relationship between hedging horizon and standard deviation](image)
Hedging with individual CDS spreads

- Hedging with individual CDS may perform a better hedge (than hedging with the index)
  - heterogeneous portfolio where some individual spreads are suddenly widening
  - equity tranche very sensitive to idiosyncratic risk
- Obviously, hedging with single name sensitivities is beyond the reach of a pure top model
- **Future research:** Comparison of hedge performance with individual CDS contracts when hedging strategies are computed
  - using the market standard hedging method (spread-deltas in a base correlation approach): **bottom-up approach**
  - using a pure top model associated with a thinning procedure: **top-down approach**
Top-down approach

- In pure top model, the flow of information is only driven by the cumulative loss process \( \mathcal{H}_t = \sigma(L_s, s \leq t) \)
  - Given \( \mathcal{H}_t \), we can only forecast the timing of defaults up to time \( t \): ordered default times are \( \mathcal{H} \)-stopping times
  - But: lose of information related to the defaulters’ identities
- Starting from a top model, Giesecke and Goldberg(2005) propose to recover single name information using a random thinning procedure
- The idea is to allocate a fraction of the loss intensity to each individual name with the constraint that the individual CDS spreads in the portfolio are matched
Set-up

- $\tau_1, \ldots, \tau_n$: default time, $N_t = \sum_{i=1}^{n} 1\{	au_i \leq t\}$
- Let us define $\mathcal{H} = \{\mathcal{H}_t\}$, where:
  \[ \mathcal{H}_t = \sigma(N_s, s \leq t) \]
- $\tau^{(1)} < \ldots < \tau^{(n)}$: ordered default time
- Let us define by $\mathcal{I} = \{\mathcal{I}_t\}$ the defaulter’s identity filtration, where
  \[ \mathcal{I}_t := \sigma(I_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, N_t) \]
- $\mathcal{F} = \{\mathcal{F}_t\}$: background filtration that contains the external market information.
- $\mathcal{G} = \{\mathcal{G}_t\}$: largest filtration
  \[ \mathcal{G}_t = \mathcal{H}_t \vee \mathcal{I}_t \vee \mathcal{F}_t \]
Compensator of ordered default times

- $\tau^{(1)} < \ldots < \tau^{(n)}$ ordered default times are $\mathcal{G}$-stopping time
- $\Lambda^{(1)}, \ldots, \Lambda^{(n)}$: $\mathcal{G}$-compensators of $\tau^{(1)}, \ldots, \tau^{(n)}$
- $\Lambda$: $\mathcal{G}$-compensator of $N$
- Bielecki, Crépey, Jeanblanc (2008):

**Proposition**

For $t \geq 0$, $\Lambda_t^{(i)} = \Lambda_{t \wedge \tau^{(i)}} - \Lambda_{t \wedge \tau^{(i-1)}}$, $i = 1, \ldots, n$

- $\tau^{(1)} < \ldots < \tau^{(n)}$ are $\mathcal{H}$-stopping time
Random Thinning

- $\tau_1, \ldots, \tau_n$ are $\mathcal{G}$-stopping times
- $\Lambda_1, \ldots, \Lambda_n$: $\mathcal{G}$-compensators of $\tau_1, \ldots, \tau_n$
- $\Lambda$: $\mathcal{G}$-compensator of $N$

$$\Lambda = \sum_{i=1}^{n} \Lambda_i$$

- Giesecke and Goldberg (2005):

**Proposition**

There exists $\mathcal{G}$-predictable non-negative processes $Z_i$, $i = 1, \ldots, n$ ($Z$-factors) such that $\sum_{i=1}^{n} Z_i = 1$ and

$$\Lambda_i = \int_0^{\cdot} Z_{i,t}d\Lambda_t, \quad i = 1, \ldots, n.$$
Random Thinning

- $\lambda_{i,t}$: $G$-intensities of $\tau_i$, $i = 1, \ldots, n$
- $\lambda_t$: $G$-intensity of $N$
- $\lambda_{i,t} = Z_{i,t} \lambda_t$, $i = 1, \ldots, n$ and $\sum_{i=1}^n Z_i = 1$
- $Z_{i,t}$ is the conditional probability that name $i$ is the next defaulter given an imminent default in the interval $[t, t + dt[$:

$$Z_{i,t} = \sum_{j=1}^n \mathbb{P} \left( \tau^{(j)} = \tau_i \mid t < \tau^{(j)} \leq t + dt, G_t \right) 1_{\{\tau^{(j-1)} < t \leq \tau^{(j)}\}}$$

- **Top-Down matrix**: $P(t) = (p_{i,j}(t))_{1 \leq i, j \leq n}$

$$p_{i,j}(t) = \mathbb{P} \left( \tau^{(j)} = \tau_i \mid t < \tau^{(j)} \leq t + dt, G_t \right)$$

- **Consistency condition**: $\sum_{i=1}^n p_{i,j}(t) = 1$, $j = 1, \ldots, n$
Random draws without replacement

- Approach proposed by Halperin and Tomecek (2008)
- TD matrix piecewise constant in $t$, only change at default times
- $t = 0$, no default $p^0_{i,j} = P(\tau^{(j)} = \tau_i)$ (inputs)
- Simulation of $\tau^{(1)}, \ldots, \tau^{(n)}$ (or $N$) in the “small filtration”, i.e $\mathcal{H}$
- At $t = \tau^{(1)}$ (first jump of $N$): independent simulation of the defaulter identity $I_1 \in \{1, 2, \ldots, n\}$ according to the distribution:
  $$P(I_1 = i) = p^0_{i,1}, \ i = 1, \ldots, n$$
- Update the TD matrix $p^0_{i,j} \rightarrow p^1_{i,j} = P(\tau^{(j)} = \tau_i \mid \tau^{(1)}, I_1)$
  $$p^1_{i,j} = \begin{cases} 
  0 & i = I_1, \ j = 1, \ldots, n \\
  0 & j = 1, \ i = 1, \ldots, n \\
  \frac{p^0_{i,j}}{1-p^0_{I,j}} & i \neq I, \ j = 2, \ldots, n 
  \end{cases}$$
- Practical issue: if $N_t = \sum_{i=1}^n 1\{\tau_i \leq t\}$ is Markov with respect to the “small filtration” $\mathcal{H}$, it is no more the case in the “large filtration” $\mathcal{G}$
Random draws with replacement

- **Pure top model**: homogeneous Markovian contagion model (local intensity)

- At the $j$-th jump of $N$: independent simulation of the defaulter identity $I_j \in \{1, 2, \ldots, n\}$ according to the distribution:

  $$p_{i,j}, \text{ where } \sum_{i=1}^{n} p_{i,j} = 1$$

- After the draw, $I_j$ is replaced in the pool: TD matrix is not updated

- Denote by $B_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$ some independent Bernoulli random variables such that $\mathbb{E}[B_{i,j}] = p_{i,j}$

- We build $n$ individual counting process $N_i(t)$ such that

  $$N_{i,t} = \sum_{j=1}^{N_t} B_{i,j}, \ i = 1, \ldots, n$$
Random draws with replacement

- $N_i$ cannot be identified with the “true” usual default process of $i$ (single jump to default)
- But here $\mathbb{E}[N_T | G_t] = \mathbb{E}[N_T | \mathcal{H}_t] = \mathbb{E}[N_T | N_t]$
- Tractable calibration to CDO tranches, individual CDS quotes
- We hope that individual delta spreads are relevant . . .