# An extension of Davis and Lo's contagion model

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MAF 2010 Fourth International Conference

Ravello, Italy, 9 April 2010

# Introduction

- Wednesday presentation by Sheri Markose illustrates the usefulness of network models to better understand systemic risk and default contagion amongst financial institutions
- Das et al. (2007) or Azizpour and Giesecke (2008) : Conditional independence assumption with no contagion effect is rejected by historical default data
- Boissay (2006), Jorion and Zhang (2007), Jorion and Zhang (2007) analyze the mechanism of default propagation and provide financial evidence of chain reactions or dominos effects
- We present a multi-period extension of Davis and Lo's contagion model

# Literature

**In the spirit of Davis and Lo's contagion model** :

- First models : Davis and Lo (2001) and Jarrow and Yu (2001)
- Extensions : Yu (2007), Egloff, Leippold and Vanini (2007), Rösch, Winterfeldt (2008), Sakata, Hisakado and Mori (2007)

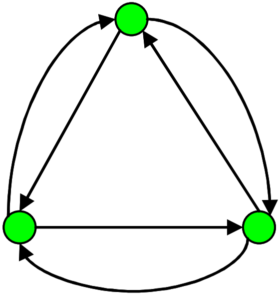
**Other contagion models in the credit risk field :**

- Copula : Schönbucher and Schubert (2001)
- Interacting particle system : Giesecke and Weber (2004)
- Incomplete information models : Frey and Runggaldier (2008), Fontana and Runggaldier (2009)
- Markov chain models : Schönbucher (2006), Frey and Backhaus (2007), Herbertsson (2007), Laurent, Cousin and Fermanian (2007)

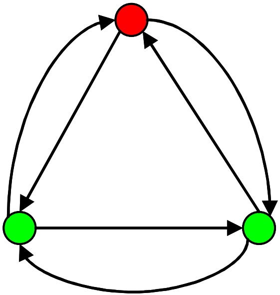
**Modeling of credit contagion for a pool of defaultable entities**

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

**Example** : Portfolio with 3 credit references

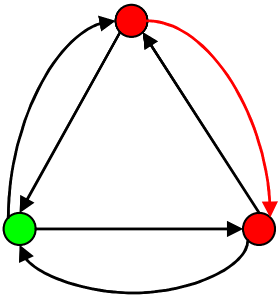


**End of the period : direct default**



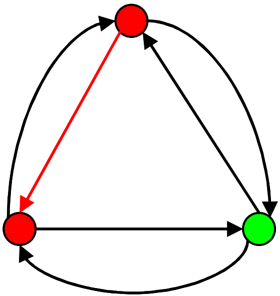
# Davis and Lo's contagion model

**End of the period : default by contagion (one possibility)**



# Davis and Lo's contagion model

**End of the period : default by contagion (another possibility)**



**One-period contagion model**

- $n$ : number of credit references
- $X_i$ : direct default indicator of name $i$.
- $C_i$ : indirect default indicator of name $i$.
- $D_i$ : default indicator (direct or indirect) such that :

$$D_i = X_i + (1 - X_i)C_i$$

where :

$$\begin{aligned} C_i &= 1 - \prod_{j \neq i} (1 - X_j Y_{ji}) \quad \text{i.e.,} \\ C_i &= \mathbb{1}_{\text{at least one } X_j Y_{ji} = 1,\ j = 1, \ldots, n} \end{aligned}$$

- $Y_{ji}$, $i, j = 1, \ldots, n$ are Bernoulli random variables
- $Y_{ji} = 1$ if the contagion link is activated from name $j$ to name $i$.

# Davis and Lo's contagion model

$N = \sum_{i=1}^{n} D_i$ : total number of defaults

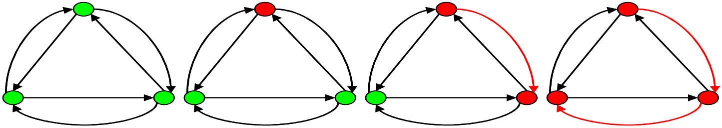
**Distribution of total number of defaults (Davis and Lo)**

$$\begin{aligned} \mathrm{P}\left[N = k\right] &= C_n^k p^k (1-p)^{n-k} (1-q)^{k(n-k)} + \\ & C_n^k \sum_{i=1}^{k-1} C_k^i p^i (1-p)^{n-i} (1-(1-q)^i)^{k-i} (1-q)^{i(n-k)}. \end{aligned}$$

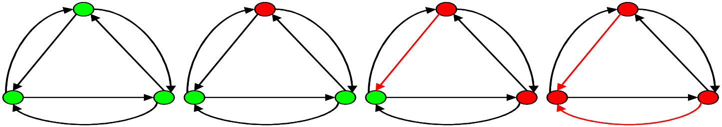
Under the assumptions :

- $X_i$, $i = 1, \ldots, n$ : **iid** Bernoulli with parameter $p$
- $Y_{ij}$, $i, j = 1, \ldots, n$ : **iid** Bernoulli with parameter $q$
- One default alone may trigger a contamination effect
- A name that has been infected cannot contaminate other names (no chain-reaction effect)

**Dominos Effect**



**Two defaults required to trigger a contagion effect**



# Extension of Davis and Lo's contagion model

**Multi-period contagion model** : $t = 0, 1, 2, \ldots, T$, in period $[t, t+1]$ :

- $n$ : number of credit references
- $X_t^i$ : direct default indicator of name $i$
- $C_t^i$ : indirect default indicator of name $i$
- $D_t^i$ : default indicator (direct or indirect) such that :

$$D_t^i = D_{t-1}^i + (1 - D_{t-1}^i)[X_t^i + (1 - X_t^i)C_t^i]$$

where

$$C_t^i = f\left(\sum_{j \in F_t} Y_t^{ji}\right)$$

- $Y_t^{ji}$, $i, j = 1, \ldots, n$ are Bernoulli random variables such that $Y_t^{ji} = 1$ if name $j$ may infect name $i$ between $t$ and $t+1$
- $F_t$ is the set of names that are likely to infect other names between $t$ and $t+1$
- $f$ is a contamination trigger function, for example $f = \mathbb{1}_{x \geq 1}$ (Davis and Lo) or $f = \mathbb{1}_{x \geq 2}$

# Extension of Davis and Lo's contagion model

$N_t = \sum_{i=1}^{n} D_t^i$ : total number of defaults at time $t$

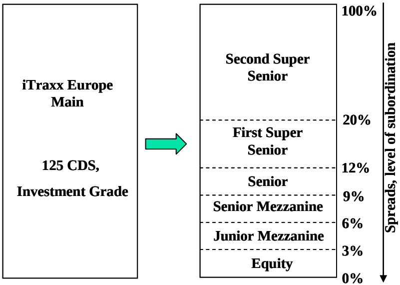
**Main result**

$$\mathrm{P}[N_t = r] = \sum_{k=0}^{r} \mathrm{P}[N_{t-1} = k] C_{n-k}^{r-k} \sum_{\gamma=0}^{r-k} C_{r-k}^{\gamma} \cdot \sum_{\alpha=0}^{n-k-\gamma} C_{n-k-\gamma}^{\alpha} \mu_{\gamma+\alpha,\, t} \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^{j+\alpha} \xi_{j+r-k-\gamma,t}(\gamma).$$

**Under the assumptions** :

- $X_t^i$, $i = 1, \ldots, n$ are conditionally independent Bernoulli random variables with the same marginal distribution and $\mathbf{X}_t = (X_t^1, ..., X_t^n)$, $t = 1, \ldots, T$ are independent vectors
- $Y_t^{j,i}$, $i, j = 1, \ldots, n$ are conditionally independent Bernoulli random variables with the same marginal distribution and $\mathbf{Y}_t = (Y_t^{j,i})_{1 \leq i,j \leq n}$, $t = 1, \ldots, T$ are independent vectors
- $(\mathbf{X}_t)_{t=1,\ldots,T}$ and $(\mathbf{Y}_t)_{t=1,\ldots,T}$ are independent

# Calibration on 5-years iTraxx tranche quotes




- Cash-flows of CDO tranches driven by the aggregate loss process

$$L_t = \sum_{i=1}^{n} (1 - R_i) D_t^i$$

where $R_i$ is the recovery rate associated with name $i$.

# Calibration on 5-years iTraxx tranche quotes

We restrict ourselves to the case where for all $t$ :

- $X_t^i \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $\mathbb{E}[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- $Y_t^{ij}$ are iid Bernoulli random variables with mean $q$, i.e., $Y_t^{ij} \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Morover

- $n = 125$, $r = 3\%$ (short-term interest rate)
- $R_i = R = 40\%$ for any $i = 1, \ldots, n$

$$L_t = (1 - R)N_t$$

- Computation of CDO tranche price only requires marginal loss distributions at several time horizons

**Least square calibration procedure** : Find $\alpha^* = (p^*, \sigma^*, q^*)$ which minimizes :

$$RMSE(\alpha) = \sqrt{\frac{1}{6}\sum_{i=1}^{6}\left(\frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i}\right)^2}.$$

where

| | 0%-3% | 3%-6% | 6%-9% | 9%-12% | 12%-20% | index |
|---|---|---|---|---|---|---|
| Market prices | $\tilde{s}_1$ | $\tilde{s}_2$ | $\tilde{s}_3$ | $\tilde{s}_4$ | $\tilde{s}_5$ | $\tilde{s}_0$ |
| model prices | $s_1(\alpha)$ | $s_2(\alpha)$ | $s_3(\alpha)$ | $s_4(\alpha)$ | $s_5(\alpha)$ | $s_0(\alpha)$ |

# Calibration on 5-years iTraxx tranche quotes

**Four calibration procedures** :

- Calibration 1 : All available market spreads are included in the fitting
- Calibration 2 : The equity [0%-3%] tranche spread is excluded
- Calibration 3 : Both equity [0%-3%] tranche and CDS index spreads are excluded
- Calibration 4 : All tranche spreads are excluded except equity tranche and CDS index spreads.

**Two calibration dates before and during the credit crisis** :

- 31 August 2005
- 31 March 2008

# Calibration on 5-years iTraxx tranche quotes

31 August 2005

| | 0%-3% | 3%-6% | 6%-9% | 9%-12% | 12%-20% | index |
|---|---|---|---|---|---|---|
| Market quotes | 24 | 81 | 27 | 15 | 9 | 36 |
| Calibration 1 | 20 | 114 | 7 | 1 | 1 | 29 |
| Calibration 2 | - | 62 | 32 | 18 | 6 | 8 |
| Calibration 3 | - | 55 | 29 | 18 | 7 | - |
| Calibration 4 | 24 | - | - | - | - | 36 |

**Annual scaled optimal parameters**

| | $p^*$ | $\sigma^*$ | $q^*$ |
|---|---|---|---|
| Calibration 1 | 0.0016 | 0.0015 | 0.0626 |
| Calibration 2 | 0.0007 | 0.0133 | 0.0400 |
| Calibration 3 | 0.0001 | 0.0025 | 0.3044 |
| Calibration 4 | 0.0014 | 0.002 | 0.1090 |

# Calibration on 5-years iTraxx tranche quotes

31 March 2008

| | 0%-3% | 3%-6% | 6%-9% | 9%-12% | 12%-20% | index |
|---|---|---|---|---|---|---|
| Market quotes | 40 | 480 | 309 | 215 | 109 | 123 |
| Calibration 1 | 28 | 607 | 361 | 228 | 95 | 75 |
| Calibration 2 | - | 505 | 330 | 228 | 112 | 68 |
| Calibration 3 | - | 478 | 309 | 215 | 109 | - |
| Calibration 4 | 40 | - | - | - | - | 123 |

**Annual scaled optimal parameters**

| | $p^*$ | $\sigma^*$ | $q^*$ |
|---|---|---|---|
| Calibration 1 | 0.0124 | 0.0886 | 0 |
| Calibration 2 | 0.0056 | 0.0518 | 0.0400 |
| Calibration 3 | 0.0012 | 0.012 | 0.2688 |
| Calibration 4 | 0.0081 | 0.0516 | 0.0589 |

# Conclusion

We propose a multi-period extension of Davis and Lo's contagion model that accounts for

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex : more than one default required to trigger a contamination)

We provide a recursive formula for the distribution of the number of defaults at different time horizons

- When direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes