Pricing and Hedging Loss Derivatives in a Markovian Bottom-Up Model with Simultaneous Defaults

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Introduction

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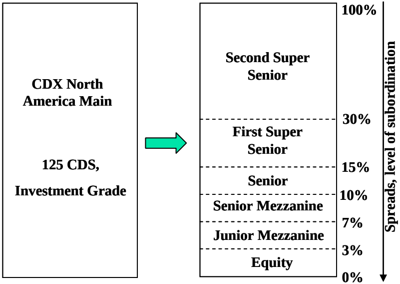
Pricing and Hedging Portfolio Credit Derivatives in a Bottom-up Model with Simultaneous Defaults

Risk management of portfolio credit derivatives



- Cash-flows driven by the realized path of the aggregate loss process

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)H_t^i$$

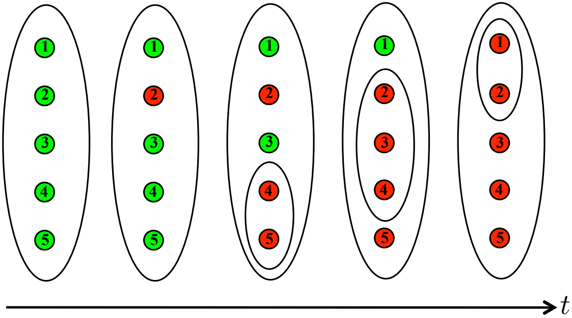
where R_i is the recovery rate and H_t^i is the default indicator of obligor i

Markovian portfolio credit risk model

Simultaneous default model

- Defaults are the consequence of trigger events affecting simultaneously pre-specified groups of obligors

Example: $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$.



Markovian portfolio credit risk model

- $\{1, \ldots, n\}$ credit references
- $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$ pre-specified groups
- $\lambda_Y(\cdot)$ intensity function associated with group $Y \in \mathcal{Y}$
- H_t^i default indicator process of name $i = 1, \ldots, n$
- $\mathbf{H}_t = (H_t^1, \ldots, H_t^n)$ is a multivariate continuous-time Markov chain in $\{0,1\}^n$ such that for $\mathbf{k}, \mathbf{m} \in \{0,1\}^n$:

$$\mathbb{P}\left(\mathbf{H}_{t+dt} = \mathbf{m} \mid \mathbf{H}_t = \mathbf{k}\right) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{\mathbf{k}^Y = \mathbf{m}\}} dt$$

  where $\mathbf{k}^Y$ is obtained from $\mathbf{k} = (k_1, \ldots, k_n)$ by replacing the components k_j, $j \in Y$, by number one. ex: $(0,1,0,0)^{\{1,2,4\}} = (1,1,0,1)$
- $\mathcal{F}_t = \sigma(\mathbf{H}_u \,, u \leq t)$ natural filtration of $\mathbf{H}$

Markovian portfolio credit risk model

Example: $n = 2$, $\mathcal{Y} = \{\{1\},\{2\},\{1,2\}\}$. $\mathbf{H}_t = (H_t^1, H_t^2)$ is a multivariate continuous-time Markov chain with space set $\{(0,0),(1,0),(0,1),(1,1)\}$ and generator matrix

$$\begin{array}{c} \\ (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \begin{pmatrix} - & \lambda_{\{1\}} & \lambda_{\{2\}} & \lambda_{\{1,2\}} \\ 0 & - & 0 & \lambda_{\{2\}} + \lambda_{\{1,2\}} \\ 0 & 0 & - & \lambda_{\{1\}} + \lambda_{\{1,2\}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- '$-$' means negative of the sum of all elements in the row
- $\lambda_{\{1\}}$: intensity function of the triggering event affecting name 1 alone
- $\lambda_{\{2\}}$: intensity function of the triggering event affecting name 2 alone
- $\lambda_{\{1,2\}}$: intensity function of the triggering event affecting the obligor group $\{1,2\}$

Markovian portfolio credit risk model

Example: $n = 2$, $\mathcal{Y} = \{\{1\},\{2\},\{1,2\}\}$. $\mathbf{H}_t = (H_t^1, H_t^2)$ is a multivariate continuous-time Markov chain with space set $\{(0,0),(1,0),(0,1),(1,1)\}$ and generator matrix

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- Obligor 1 defaults with intensity $\lambda_{\{1\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool
- Obligor 2 defaults with intensity $\lambda_{\{2\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool

General case: Obligor i defaults with intensity $\eta_i(t) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{i \in Y\}}$

No contagion effect

Pricing and Hedging

Markov copula condition

For any $i = 1, \dots, n$, H^i is a one dimensional Markov process with respect to the global filtration $\mathcal{F}$:

$$\mathbb{E}\left[\Phi(H^i_T) \mid \mathcal{F}_t\right] = \mathbb{E}\left[\Phi(H^i_T) \mid H^1_t, \dots, H^n_t\right] = \mathbb{E}\left[\Phi(H^i_T) \mid H^i_t\right]$$

Independent pricing and calibration of single-name products

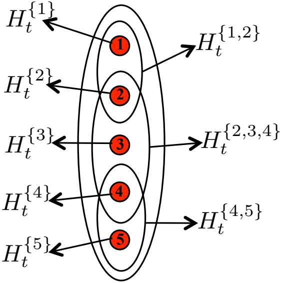
Hedging CDO tranches with single-name CDS

- Dynamics of CDO tranche prices and single-name CDS prices can be expressed in terms of some fundamental martingales
- Price of portfolio loss derivatives solves the Kolmogorov backward equations
- Computation of min-variance hedging strategies

Numerically intractable at least for large heterogeneous portfolios ($n > 20$)

Common Shocks Model Interpretation

Example: $n = 5$ and $\mathcal{Y} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{4,5\},\{2,3,4\},\{1,2\}\}$.



$$\hat{H}_t^1 := \max\left\{H_t^{\{1\}}, H_t^{\{1,2\}}\right\}$$
$$\hat{H}_t^2 := \max\left\{H_t^{\{2\}}, H_t^{\{1,2\}}, H_t^{\{2,3,4\}}\right\}$$
$$\hat{H}_t^3 := \max\left\{H_t^{\{3\}}, H_t^{\{2,3,4\}}\right\}$$
$$\hat{H}_t^4 := \max\left\{H_t^{\{4\}}, H_t^{\{2,3,4\}}, H_t^{\{4,5\}}\right\}$$
$$\hat{H}_t^5 := \max\left\{H_t^{\{5\}}, H_t^{\{4,5\}}\right\}$$

General case: In the common shocks model, the individual default indicators are such that

$$\hat{H}_t^i := \max\left\{H_t^Y, Y \in \mathcal{Y}, i \in Y\right\}$$

where H_t^Y, $Y \in \mathcal{Y}$ are independent $\{0,1\}$–point processes with intensity λ_Y:

$$\mathbb{P}(H_t^Y = 0) = \exp\left(-\int_0^t \lambda_Y(u)du\right)$$

Common Shocks Model Interpretation

Main result

- $\hat{\tau}_i := \inf\left\{t \geq 0 \mid \hat{H}_t^i = 1\right\}$, $i = 1, \ldots, n$, default times in the common shocks model
- $\tau_i := \inf\left\{t \geq 0 \mid H_t^i = 1\right\}$, $i = 1, \ldots, n$, default times in the Markovian model

Proposition

For all $t_1, \ldots, t_n \geq 0$, the following relation holds

$$\mathbb{P}\left(\hat{\tau}_1 > t_1, \ldots, \hat{\tau}_n > t_n\right) = \mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n\right)$$

Common Shocks Model Interpretation

Main result (conditional version)

- Supp$(\mathbf{H}_t)$: set of all defaulted names at time t
- $\mathcal{Y}_t = \{Y \in \mathcal{Y} \,;\, Y \nsubseteq \text{Supp}(\mathbf{H}_t)\}$: set of all groups that contain at least one alive obligor
- $\hat{H}^i_t := \max\left\{H^Y_t,\, Y \in \mathcal{Y}_t,\, i \in Y\right\}$: individual default processes in the time-t conditional common-shocks model
- $\hat{\tau}_i := \inf\left\{\theta \geq t \mid \hat{H}^i_\theta = 1\right\}$, $i \in (\text{Supp}(\mathbf{H}_t))^c$, default times in the common-shock model for names that have survived up to time t
- $\tau_i := \inf\left\{\theta \geq t \mid H^i_\theta = 1\right\}$, $i \in (\text{Supp}(\mathbf{H}_t))^c$, default times in the Markovian model for names that have survived up to time t

Proposition

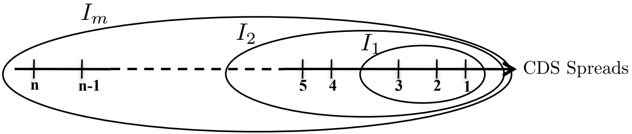
For all $t_1, \ldots, t_n \geq t$, the following relation holds

$$\mathbb{P}\left(\hat{\tau}_1 > t_1,\, i \in (\text{Supp}(\mathbf{H}_t))^c \mid \mathcal{F}_t\right) = \mathbb{P}\left(\tau_i > t_i,\, i \in (\text{Supp}(\mathbf{H}_t))^c \mid \mathcal{F}_t\right)$$

Common Shocks Model Interpretation

Calibration of individual intensities on single-name CDS

- Individual shocks + Common shocks: $\mathcal{Y} = \{\{1\}, \dots, \{n\}, I_1, \dots, I_m\}$
- Names are ordered with respect to riskiness



- Price of CDS i can be expressed as a function of $\mathbb{E}\left[H_t^i\right]$, $t = 0, \dots, T$

$$\mathbb{E}\left[H_t^i\right] = 1 - \exp\left(-\int_0^t \eta_i(u)dt\right)$$

where

$$\eta_i(u) = \lambda_{\{i\}}(u) + \sum_{k=1}^{m} \lambda_{I_k}(u)\mathbf{1}_{\{i \in I_k\}}$$

- η_i, $i = 1, \dots, n$ calibrated on individual CDS curves using a bootstrap procedure

Common Shocks Model Interpretation

Calibration of common-shocks intensities on CDO tranches

- Pricing of CDO tranches only involves marginal loss distributions
- Thanks to the common-shock model interpretation:

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1-R_i)H_t^i \stackrel{d}{=} \frac{1}{n}\sum_{i=1}^{n}(1-R_i)\hat{H}_t^i$$

- Conditionally on $\left(H_t^{I_1}, \ldots, H_t^{I_m}\right)$, $\hat{H}^1, \ldots, \hat{H}^n$ are independent Bernoulli random variables with parameters

$$p_t^i = \begin{cases} 1 & i \in \cup_{k=1}^m \{I_k \,;\, H_t^{I_k} = 1\} \\ 1 - \exp\left(-\int_0^t \lambda_{\{i\}}(u)du\right) & \text{else} \end{cases}$$

where

$$\lambda_{\{i\}}(u) = \eta_i(u) - \sum_{k=1}^{m} \lambda_{I_k}(u)\mathbf{1}_{\{i \in I_k\}} \geq 0$$

Common Shocks Model Interpretation

Fast convolution-recursion procedure for computing loss distribution

- Let $N_t^{(k)} = \sum_{i=1}^{k} \hat{H}_t^i$, $k = 1, \ldots, n$
- Let $q_t^{(k)}(i) = \mathbb{P}\left(N_t^{(k)} = i \mid H_t^{I_1}, \ldots, H_t^{I_m}\right)$, $i = 0, \ldots, k$
- The following recursion procedure can be used to compute the conditional loss distribution starting from $k = 0$ and $q_t^{(0)}(0) = 1$

$$\begin{cases} q_t^{(k+1)}(0) = (1 - p_t^{k+1}) \cdot q_t^{(k)}(0) \\ q_t^{(k+1)}(i) = p_t^{k+1} \cdot q_t^{(k)}(i-1) + (1 - p_t^{k+1}) \cdot q_t^{(k)}(i), \quad i = 1, \ldots, k \\ q_t^{(k+1)}(k+1) = p_t^{k+1} \cdot q_t^{(k)}(k) \end{cases}$$

- This gives the time-t conditional distribution $q_t^{(n)}$ of the total number of defaults $N_t = N_t^{(n)} = \sum_{i=1}^{n} \hat{H}_t^i$

Common Shocks Model Interpretation

Fast convolution-recursion procedure for computing loss distribution

- One can remark that $\Omega = \bigcup_{k=1}^{m} A_t^k$ where

$$\begin{cases} A_t^0 = \left\{ H_t^{I_1} = 0, \ldots, H_t^{I_m} = 0 \right\} \\ A_t^k = \left\{ H_t^{I_k} = 1, H_t^{I_{k+1}} = 0, \ldots, H_t^{I_m} = 0 \right\}, \quad k = 1, \ldots, m-1 \\ A_t^m = \left\{ H_t^{I_m} = 1 \right\} \end{cases}$$

- For every $k = 1, \ldots, m$, $\hat{H}_t^1, \ldots, \hat{H}_t^n$ are conditionally independent Bernoulli given A_t^k. Since A_t^k, $k = 1, \ldots, m$ are disjoint events

$$\mathbb{P}(N_t = i) = \sum_{k=1}^{m} \mathbb{P}(N_t = i \mid A_t^k) \mathbb{P}(A_t^k), \; i = 0, \ldots, n$$

- $\mathbb{P}(N_t = i \mid A_t^k)$ can be computed thanks to the previous recursion procedure
- As $H_t^{I_k}$, $k = 1, \ldots, m$ are independent, the probability of the event A_t^k satisfies

$$\mathbb{P}(A_t^k) = \left(1 - \exp\left(-\int_0^t \lambda_{I_k}(u)du\right)\right) \prod_{j=k+1}^{m} \exp\left(-\int_0^t \lambda_{I_j}(u)du\right)$$

Calibration on CDX index

Data set: 5-years CDX North-America IG index on 20 December 2007

- Quoted spreads (at different pillars) of the 125 index constituents
- Quoted spreads of standard tranches [0,3], [3,7], [7,10], [10,15], [15,30]

Model specification:

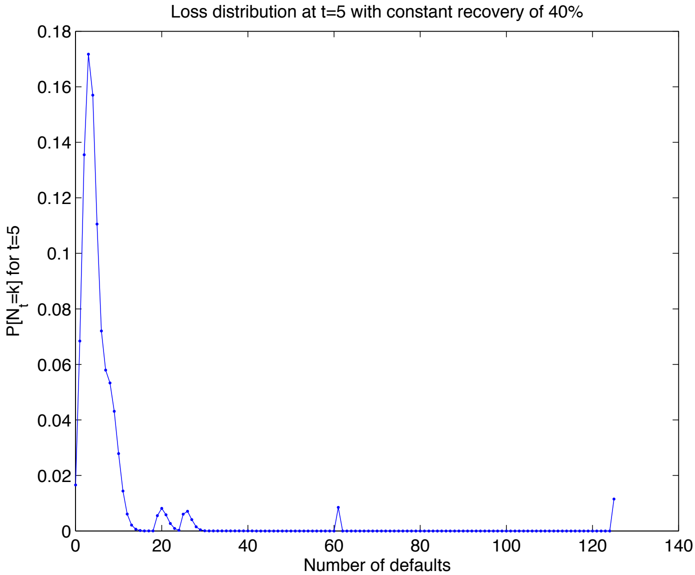
- 5 groups $I_1 \subset \cdots \subset I_5$ such that $I_1 = \{1, \ldots, 6\}$, $I_2 = \{1, \ldots, 19\}$, $I_3 = \{1, \ldots, 25\}$, $I_4 = \{1, \ldots, 61\}$, $I_5 = \{1, \ldots, 125\}$
- Piecewise constant intensities $\lambda_{\{1\}}, \ldots, \lambda_{\{125\}}$, $\lambda_{I_1}, \ldots, \lambda_{I_5}$ with grid points corresponding to CDS pillars
- Homogeneous recovery rate across obligors: 40%
- Interest rate: 3%

Calibration results:

Tranche	[0,3]	[3,7]	[7,10]	[10,15]	[15,30]
Model spread in bps	48.0701	254.0000	124.0000	61.0000	38.9390
Market spread in bps	48.0700	254.0000	124.0000	61.0000	41.0000
Abs. Err. in bps	0.0001	0.0000	0.0000	0.0000	2.0610
% Rel. Err.	0.0001	0.0000	0.0000	0.0000	5.0269

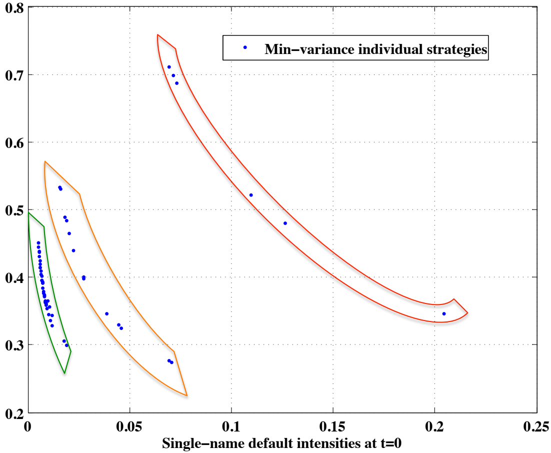
Calibration on CDX index

5-years calibrated loss distribution:



Min-variance hedging strategies

- **Aim**: hedging the [0-3%] equity tranche with one particular CDS in the index
- Min-variance hedging strategies associated with the 61 riskiest CDS



Thank you for your attention!

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