Some Proposals about Bivariate Risk Measures

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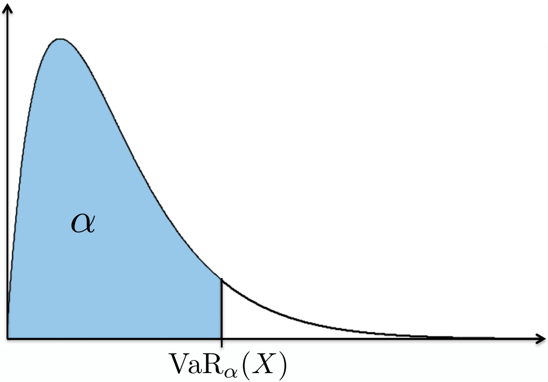
- Under Basel II or Solvency II, each financial institution computes its own regulatory capital in a methodology that does not include risks undertaken by the other institutions even if the latter may be highly interconnected. (“micro-prudential regulation”)

- Risks cannot be diversify away among different institutions. Recent interest for a macro-prudential regulation with an helicopter view on the whole financial system.

- How could we construct a capital rule that reflect both the individual risks and interconnection among these risks in a situation where we cannot benefit from diversification ?

Value-at-Risk paradigm



Given an univariate continuous and strictly monotonic loss distribution function F_X,

$$\mathrm{VaR}_\alpha(X) = Q_X(\alpha) = F_X^{-1}(\alpha), \quad \forall\, \alpha \in (0,1).$$

Shortcoming of VaR measure:

- VaR does not give any information on the severity of loss when larger than the VaR
- VaR is not a coherent risk measure (see Artzner, 1999)

To overcome problems of VaR → Conditional-Tail-Expectation (CTE):

$$CTE_\alpha(X) = \mathbb{E}[\,X \,|\, X \geq \mathrm{VaR}_\alpha(X)\,] = \mathbb{E}[\,X \,|\, X \geq Q_X(\alpha)\,],$$

Dependence and dimensional problems

Riskiness not only of the marginal distributions, but also of the joint distribution:

$$\rho : \quad \mathbf{X} := (X_1, \ldots, X_d) \mapsto \begin{pmatrix} \rho^1[\mathbf{X}] \\ \vdots \\ \rho^d[\mathbf{X}] \end{pmatrix} \in \mathbb{R}_+^d,$$

Risk measures essentially based on a "*distributional approach*" (i.e. we have to capture the information coming both from the marginal distributions and from the dependence structure).

Multivariate *Value-at-Risk* as quantile curve (Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009), i.e., the set

$$\partial L(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d \, : \, F(\mathbf{x}) = \alpha\}$$

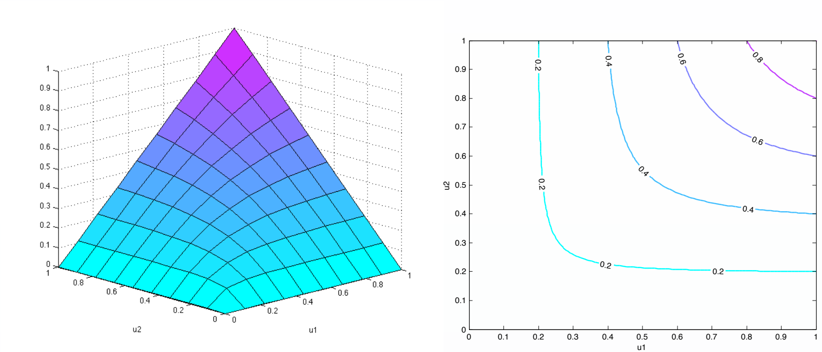


Figure: **left**: cdf of a Clayton copula with parameter 2, **right**: a set of associated quantile curves

A multivariate *Value-at-Risk* and *Conditional-Tail-Expectation*

Definition

Consider a random vector $\mathbf{X}$ with absolutely continuous cdf F. For $\alpha \in (0,1)$, we define:

$$\mathrm{VaR}_\alpha(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \mathbf{X} \in \partial L(\alpha) \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \mathbf{X} \in \partial L(\alpha) \,] \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) = \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) = \alpha \,] \end{pmatrix},$$

$$\mathrm{CTE}_\alpha(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, \mathbf{X} \in L(\alpha) \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, \mathbf{X} \in L(\alpha) \,] \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\, X_1 \,|\, F(\mathbf{X}) \geq \alpha \,] \\ \vdots \\ \mathbb{E}[\, X_d \,|\, F(\mathbf{X}) \geq \alpha \,] \end{pmatrix},$$

where $\partial L(\alpha)$ is the α-level set of F and $L(\alpha)$ is the upper α-level set of F .

Example: Bivariate Archimedean copula case

$$\mathrm{VaR}^1_\alpha(X,Y) = \frac{\int_{Q_{\mathbf{X}}(\alpha)}^{\infty} x\, f_{(U,C(U,V))}(F_X(x),\alpha)\,\mathrm{d}x}{K'(\alpha)},$$

where $f_{(U,C(U,V))}$ is the density of the cdf $F_{(U,C(U,V))}$ given by

$$F_{(U,C(U,V))}(s,t) = t - \frac{\phi(t)}{\phi'(t)} + \frac{\phi(s)}{\phi'(t)}, \qquad \text{for } 0 < t < s < 1.$$

and K is the cdf of $C(U,V)$ (Kendall distribution)

Copula	θ	$\mathrm{VaR}^1_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1,\infty)$	$\frac{\theta}{\theta-1}\frac{\alpha^\theta-\alpha}{\alpha^\theta-1}$
Counter-monotonic W	-1	$\frac{1+\alpha}{2}$
Independent Π	0	$\frac{\alpha-1}{\ln\alpha}$
Comonotonic M	∞	α

Example: Bivariate Clayton copula case

Remark: for Clayton $\frac{\partial \mathrm{VaR}^{\mathbf{1}}_{\alpha,\theta}}{\partial \alpha} \geq 0$ and $\frac{\partial \mathrm{VaR}^{\mathbf{1}}_{\alpha,\theta}}{\partial \theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0,1)$.

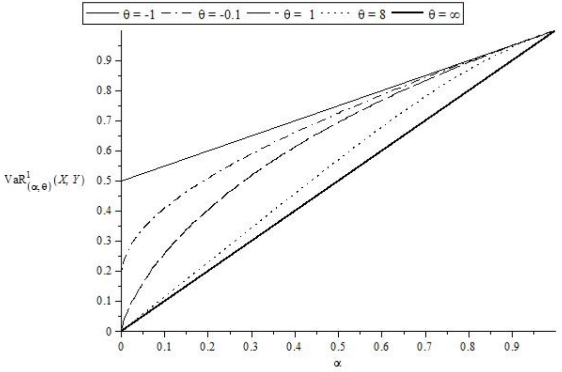


Figure: Behavior of $\mathrm{VaR}^1_{\alpha,\theta}(X,Y) = \mathrm{VaR}^2_{\alpha,\theta}(X,Y)$ with respect to risk level α for different values of dependence parameter θ. The random vector (X,Y) follows a Clayton copula distribution with parameter θ.

Example: Bivariate Clayton copula case

Copula	θ	$\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$
Clayton C_θ	$(-1,\infty)$	$\frac{1}{2}\,\frac{\theta}{\theta-1}\,\frac{\theta-1-\alpha^2(1+\theta)+2\,\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic W	-1	$\frac{1}{4}\,\frac{1-\alpha^2+2\ln\alpha}{1-\alpha+\ln\alpha}$
Independent Π	0	$\frac{1}{2}\,\frac{(1-\alpha)^2}{1-\alpha+\alpha\ln\alpha}$
Comonotonic M	∞	$\frac{1+\alpha}{2}$

Table: $\mathrm{CTE}^1_{\alpha,\theta}(X,Y)$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\alpha} \geq 0$ and $\frac{\partial \mathrm{CTE}^1_{\alpha,\theta}}{\partial\theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0,1)$.

Example: Bivariate Frank copula case

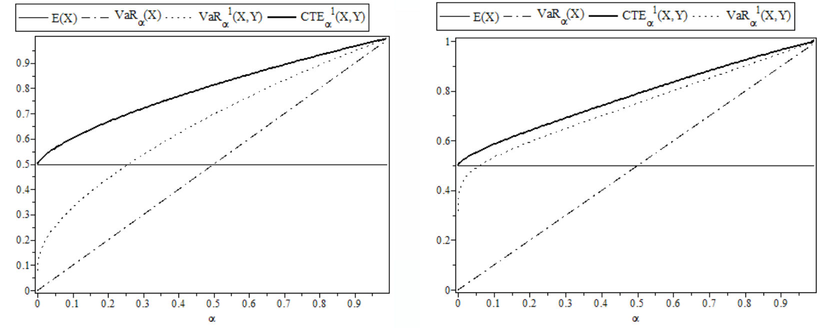


Figure: Frank copula with standard uniform marginals, parameter $\theta = 2$ (left), parameter $\theta = -10$ (right).

	$\mathrm{VaR}_\alpha(\mathbf{X})$	$\mathrm{CTE}_\alpha(\mathbf{X})$
Several (axiomatic) properties	Invariance properties ($\mathbf{c} \in \mathbb{R}_+^d$): • $\mathrm{VaR}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\mathrm{VaR}_\alpha(\mathbf{X})$, • $\mathrm{VaR}_\alpha(\mathbf{c}+\mathbf{X}) = \mathbf{c}+\mathrm{VaR}_\alpha(\mathbf{X})$. Lower bounds: • $\mathrm{VaR}^i_\alpha(\mathbf{X}) \geq \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. Analytical closed-form formulas for $\mathrm{VaR}_\alpha(\mathbf{X})$ and $\mathrm{CTE}_\alpha(\mathbf{X})$.	Invariance properties ($\mathbf{c} \in \mathbb{R}_+^d$): • $\mathrm{CTE}_\alpha(\mathbf{c}\,\mathbf{X}) = \mathbf{c}\,\mathrm{CTE}_\alpha(\mathbf{X})$, • $\mathrm{CTE}_\alpha(\mathbf{c}+\mathbf{X}) = \mathbf{c}+\mathrm{CTE}_\alpha(\mathbf{X})$. Lower bounds: • $\mathrm{CTE}^i_\alpha(\mathbf{X}) \geq \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. Safety loading: • $\mathrm{CTE}^i_\alpha(\mathbf{X}) \geq \mathbb{E}[X_i]$ • $\mathrm{CTE}_0(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$.
Risk level	$\mathrm{VaR}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α.	• $\mathrm{CTE}^i_\alpha(\mathbf{X}) \geq \mathrm{VaR}^i_\alpha(\mathbf{X})$, $\forall\, \alpha \in (0,1)$. • $\mathrm{CTE}^i_\alpha(\mathbf{X})$ is a non-decreasing function of α.

- ✓ These two risk measures both satisfy the positive homogeneity and the translation invariance property (Artzner *et al.*, 1999).
- ✓ Comparison results between univariate risk measures and components of multivariate risk measures are provided.
- ✓ Change in risk level α.

	$\mathrm{VaR}_\alpha(\mathbf{X})$	$\mathrm{CTE}_\alpha(\mathbf{X})$
Dependence structure	Comonotonic case: • $\mathrm{VaR}^{i}_{\alpha}(\mathbf{X}) = \mathrm{VaR}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. For a fixed copula C and $X_i \overset{d}{=} Y_i$: • $\mathrm{VaR}^{i}_{\alpha}(\mathbf{X}) = \mathrm{VaR}^{i}_{\alpha}(\mathbf{Y})$, $\forall\, \alpha \in (0,1)$. For a fixed copula C and $X_i \preceq_{st} Y_i$: • $\mathrm{VaR}^{i}_{\alpha}(\mathbf{X}) \leq \mathrm{VaR}^{i}_{\alpha}(\mathbf{Y})$, $\forall\, \alpha \in (0,1)$.	Comonotonic case: • $\mathrm{CTE}^{i}_{\alpha}(\mathbf{X}) = \mathrm{CTE}_\alpha(X_i)$, $\forall\, \alpha \in (0,1)$. For a fixed copula C and $X_i \overset{d}{=} Y_i$: • $\mathrm{CTE}^{i}_{\alpha}(\mathbf{X}) = \mathrm{CTE}^{i}_{\alpha}(\mathbf{Y})$, $\forall\, \alpha \in (0,1)$. For a fixed copula C and $X_i \preceq_{D} Y_i$: • $\mathrm{CTE}^{i}_{\alpha}(\mathbf{X}) \leq \mathrm{CTE}^{i}_{\alpha}(\mathbf{Y})$, $\forall\, \alpha \in (0,1)$.

- ✓ Change in marginal distributions and in dependence structure.
- ✓ Results turn to be consistent with existing properties on univariate risk measures.
- ✓ $\theta \leq \theta^* \Rightarrow \mathrm{VaR}^1_\alpha(X^*, Y^*) \leq \mathrm{VaR}^1_\alpha(X, Y)$ (Archimedean copula family).

Perspectives

A. Cousin, E. Di Bernardino, *A multivariate extension of Value-at-Risk and Conditional-Tail-Expectation*, submitted to *Journal of Multivariate Analysis* (2011), http://hal.archives-ouvertes.fr/hal-00638382/fr/.

- ✓ Comparisons of our multivariate CTE and VaR with existing multivariate generalizations of these measures, both theoretically and experimentally: applications on financial portfolios; micro-prudential versus macro-prudential approach, ...
- ✓ Extension to the case of discrete distribution functions.

Thank you for your attention