An Extension of Davis and Lo's Contagion Model

Areski Cousin
Joint work with Diana Dorobantu and Didier Rullière
ISFA, University of Lyon [1]

5th Financial Risks International Forum, Paris, 22 March 2012

1. This work has been funded by ANR Research Project ANR-08-BLAN-0314-01 and by Chaire BNP-Paribas Assurance "management de la modélisation"

Introduction

Empirical studies on contagion mechanisms

- Das, Duffie, Kapadia, Saita (2007) : Conditional independence assumption with no contagion effect is rejected by historical default data. The conditional independence assumption is not enough to fully capture the observed clustering in default events
- Boissay (2006), Jorion and Zhang (2007, 2009) analyze the mechanism of default propagation and provide financial evidence of chain reactions or dominos effects

Need for a dynamic model with defaults dependencies and contagion

- Default risks may be connected to underlying macro-economic factors
- Contagion mechanisms
- Chain reactions and evolution over time

Literature

Some contagion models in the credit risk field

- Intensities depending on defaults : Jarrow and Yu (2001), Yu (2007)
- Markov chain models : Schönbucher (2006), Frey and Backhaus (2007), Herbertsson (2007), Laurent, Cousin and Fermanian (2007)
- Copula : Schönbucher and Schubert (2001)
- Incomplete information models : Giesecke (2004), Frey and Runggaldier (2008), Fontana and Runggaldier (2009), Frey and Schmidt (2009)

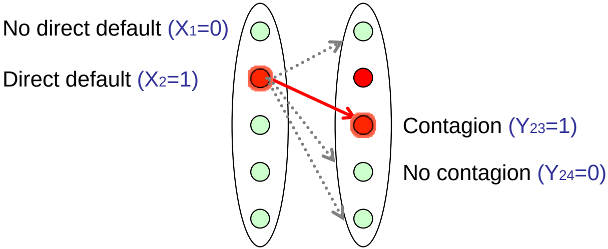
In the spirit of Davis and Lo's contagion model

- First models : Davis and Lo (2001)
- Extensions : Sakata, Hisakado and Mori (2007), Egloff, Leippold and Vanini (2007), Rösch, Winterfeldt (2008)
- We propose a multiperiod extension of Davis and Lo's contagion model.

Modeling of credit contagion for a pool of defaultable entities

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

Example : Portfolio with 5 credit references over one period



Davis and Lo's contagion model

One-period contagion model

- n : number of credit references,
- X_i : direct default indicator of name i (i.e. $X_i = 1$ if i defaults directly, $X_i = 0$ otherwise),
- $Y_{ji} = 1$ if the contagion link is activated from name j to name i, $Y_{ji} = 0$ otherwise.

- $\mathscr{C}_i$: indirect default indicator of name i,
- Z_i : resulting default indicator (direct or indirect) such that :

$$Z_i = X_i + (1 - X_i)\mathscr{C}_i$$

where :

$$\begin{aligned} \mathscr{C}_i &= \mathbb{1}_{\text{at least one } X_j Y_{ji}=1,\ j=1,\dots,n} \\ &= 1 - \prod_{j \neq i} (1 - X_j Y_{ji}) \end{aligned}$$

Davis and Lo's contagion model

$N = \sum_{i=1}^{n} Z_i$: total number of defaults

Distribution of total number of defaults (Davis and Lo)

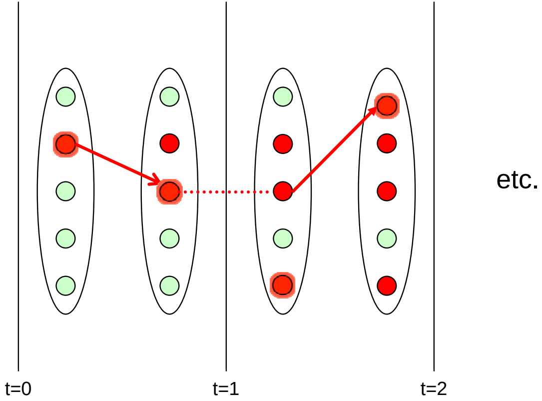
$$\mathrm{P}\left[N = k\right] = C_n^k p^k (1-p)^{n-k} (1-q)^{k(n-k)} + C_n^k \sum_{i=1}^{k-1} C_k^i p^i (1-p)^{n-i} (1-(1-q)^i)^{k-i} (1-q)^{i(n-k)}.$$

Under the assumptions :

- Direct defaults X_i, $i = 1, \ldots, n$: iid Bernoulli with parameter p
- Contagion links Y_{ij}, $i, j = 1, \ldots, n$: iid Bernoulli with parameter q
- One contagion link alone may trigger an indirect default
- Infected entities cannot contaminate others (no chain-reaction effect)

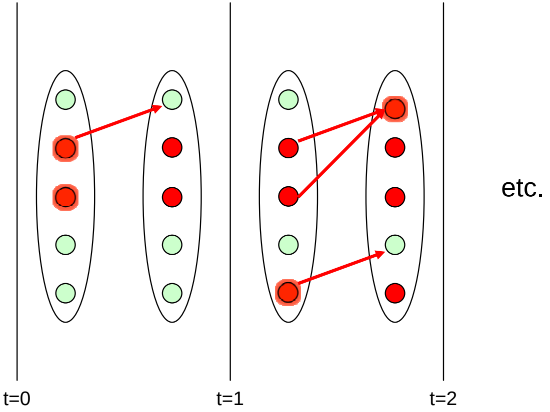
Dominos Effect

- The model becomes a multiperiod model
- One can choose the set of entities likely to contaminate others
- some iid assumptions are released



Contagion incidence on indirect default

- One can change the number of contagion links required to cause a default (here two contaminations required)



Extension of Davis and Lo's contagion model

Multi-period contagion model : $t = 1, 2, \ldots, T$, in period $[t-1, t]$:

- n : number of credit references,
- X_t^i : direct default indicator of entity i,
- Y_t^{ji} : contagion links are Bernoulli random variables such that $Y_t^{ji} = 1$ if entity j may infect entity i,
- Z_t^i : resulting default indicator (direct or indirect) such that :

$$Z_t^i = Z_{t-1}^i + (1 - Z_{t-1}^i)[X_t^i + (1 - X_t^i)\mathscr{C}_t^i]$$

- $\mathscr{C}_t^i = f\left(\sum_{j \in F_t} Y_t^{ji}\right)$: indirect default indicator of name i,
- F_t is the set of names that are likely to infect other names between t and $t+1$
- f is a contamination trigger function, for example $f = \mathbb{1}_{x \geq 1}$ (Davis and Lo) or $f = \mathbb{1}_{x \geq 2}$

Extension of Davis and Lo's contagion model

$N_t = \sum_{i=1}^{n} Z_t^i$: total number of defaults at time t

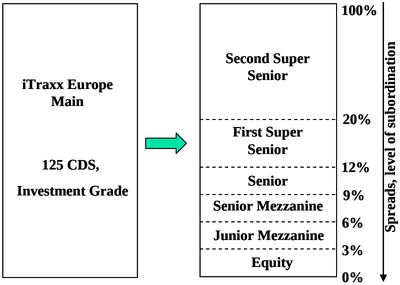
Main result

$$\mathrm{P}[N_t = r] = \sum_{k=0}^{r} \mathrm{P}[N_{t-1} = k] C_{n-k}^{r-k} \sum_{\gamma=0}^{r-k} C_{r-k}^{\gamma} \cdot \sum_{\alpha=0}^{n-k-\gamma} C_{n-k-\gamma}^{\alpha} \mu_{\gamma+\alpha,\,t} \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^{j+\alpha} \xi_{j+r-k-\gamma,t}(\gamma).$$

Under the assumptions :

- Direct defaults X_t^i, $i = 1, \ldots, n$ are conditionally independent Bernoulli r.v. with the same marginal distribution and $\mathbf{X}_t = (X_t^1, ..., X_t^n)$, $t = 1, \ldots, T$ are independent vectors.
- Contagion links Y_t^{ji}, $i, j = 1, \ldots, n$ are conditionally independent Bernoulli r.v. with the same marginal distribution and $\mathbf{Y}_t = (Y_t^{ji})_{1 \leq i,j \leq n}$, $t = 1, \ldots, T$ are independent vectors.
- $(\mathbf{X}_t)_{t=1,\ldots,T}$ and $(\mathbf{Y}_t)_{t=1,\ldots,T}$ are independent.

Calibration on 5-years iTraxx tranche quotes



- Cash-flows of CDO tranches driven by the aggregate loss process (in %)

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)Z_t^i$$

where R_i is the recovery rate associated with name i.

- if $R_i = R$ for any $i = 1, \dots, n$

$$L_t = \frac{1}{n}(1 - R) \cdot N_t$$

Calibration on 5-years iTraxx tranche quotes

We restrict ourselves to the case where for all t :

- Direct defaults $X_t^i \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $\mathbb{E}[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- Contagion links Y_t^{ij} are iid $Y_t^{ij} \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Moreover

- $n = 125$, $r = 3\%$ (short-term interest rate)
- Recovery rate $R = 40\%$
- Computation of CDO tranche price only requires marginal loss distributions at several time horizons

Least square calibration procedure : Find $\alpha^* = (p^*, \sigma^*, q^*, R^*)$ which minimizes :

$$RMSE(\alpha) = \sqrt{\frac{1}{6}\sum_{i=1}^{6}\left(\frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i}\right)^2}.$$

where

	0%-3%	3%-6%	6%-9%	9%-12%	12%-20%	index
Market prices	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{s}_4$	$\tilde{s}_5$	$\tilde{s}_0$
model prices	$s_1(\alpha)$	$s_2(\alpha)$	$s_3(\alpha)$	$s_4(\alpha)$	$s_5(\alpha)$	$s_0(\alpha)$

To improve the results we consider :

- One additional external contagious entity

	0%-3%	3%-6%	6%-9%	9%-12%	12%-20%	index	RMSE
31 Jan 2008							
Market spreads	31	317	212	140	74	77	-
Model spreads	32	328	204	142	77	64	7.5
1st Mar 2007							
Market spreads	10	46	13	6	2	23	-
Model spreads	10	37	14	6	2	21	9.2

Table: iTraxx Europe main market and model spreads (in bp) and the corresponding root mean square errors. The [0%-3%] spread is quoted in %. All maturities are for five years.

corresponding optimal parameters (on quarterly periods)

	p^*	σ_X^*	q^*	R^*
31 Jan 2008	0.0012	0.0151	0.0007	0.1964
1st Mar 2007	0.0001	0.0026	0.0005	0.1346

Table: Optimal parameters $\alpha^* = (p^*, \sigma_X^*, q^*, R^*)$.

Conclusion

We propose a multi-period extension of Davis and Lo's contagion model that accounts for

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex : more than one default required to trigger a contamination)

We provide a recursive formula for the distribution of the number of defaults at different time horizons

- especially when direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes

Thank you for your attention.

Similar kind of formulas hold when we have :

finite-exchangeability

- Direct defaults may be finite-exchangeable (does not imply conditional independence as infinite exchangeability, De Finetti's Theorem does not apply here).

non stationarity

- Joint law for *Direct defaults* and for *Contagion links* may change over time.

heterogeneity (with higher complexity)

- Direct defaults may be dependent and heterogeneous, in a monoperiodic framework.
- Direct defaults may be dependent and heterogeneous, in a multiperiodic framework, but with an exponential complexity (need to consider all possible sets of remaining entities at time t).

Probabilistic tools

Waring's Formula - special case of Schuette-Nesbitt Formula

If $B^1, ..., B^n$ are n dependent Bernoulli r.v. and $\Gamma \subset \{1, \ldots, n\}$ with cardinal m,

$$\mathrm{P}\left[\sum_{i\in\Gamma} B^i = k\right] = \mathbb{1}_{k\leq m} C_m^k \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \mu_{j+k}(\Gamma).$$

$$\text{with } \mu_k(\Gamma) = \frac{1}{C_m^k} \sum_{\substack{j_1<j_2<..<j_k \\ j_1,\ldots,j_k\in\Gamma}} \mathrm{P}\left[B^{j_1}=1\cap\ldots\cap B^{j_k}=1\right], \quad k\geq 1,$$

coefficients μ_k may be simplified :

- if independence (without requiring iid) : products
- if exchangeability : the sum vanishes

Here we are looking for :

- Directs defaults : $\sum_{j\in\Gamma} X_t^j$ as a function of some coefficients $\mu_{k,t}(\Gamma)$,
- Contagion links : $\sum_{j\in F_t} Y_t^{\sigma(j)}$ as a function of some coefficients $\lambda_{k,t}$,
- Indirects defaults : $\sum_{j=1\ldots k} \mathscr{C}_t^j$ as a function of some coefficients $\xi_{k,t}$,

Appendix I - probabilistic tools

Appendix I - Probabilistic tools

Infinite- exchangeability

$A_1, A_2, \ldots$ sequence of exchangeable r.v. if for all n and for any permutation σ

$$A_1, \ldots, A_n \stackrel{\mathcal{D}}{=} A_{\sigma(1)}, \ldots, A_{\sigma(n)} ,$$

De Finetti's Theorem

$A_1, A_2, \ldots$ is a sequence of infinite-exchangeable Bernoulli r.v.
iff there exist a r.v. $\Theta \in [0, 1]$ such that, conditionally to Θ
$A_1, A_2, \ldots$ is an iid sequence of Bernoulli r.v. with parameter Θ

- Here, calculations given Θ but difficulties to simplify
- De Finetti's Theorem does not apply for finite-exchangeability
- Need for other tools

Appendix I - Probabilistic tools

If N is a number of fulfilled events B_i, $i \in \Omega$,
A linear combination of $\mathrm{P}[N = k]$ will be written :

Schuette-Nesbitt formula

$$\sum_{k \in \Omega} \mathrm{P}[\mathbf{N} = \mathbf{k}] f(k) = \sum_{k \in \Omega} \mathbf{S}_\mathbf{k} \Delta^k f(0)$$

$$\text{avec } S_k = \sum_{j_1 < \ldots < j_k} \mathrm{P}\left[B_{j_1} \cap \cdots \cap B_{j_k}\right]$$

$$\Delta f(k) = f(k+1) - f(k), \text{ difference operator}$$

- events of kind $[N = k]$ given coefficients S_k.
- S_k can be simplified with independence, without requiring i.i.d.
- S_k can be simplified with exchangeability
- events of kind $[N = k]$ as simple as $[N = 0]$ or $[N \geqslant 1]$

Appendix I - Probabilistic tools

In the particular case where $f(j) = \mathbb{1}_{j=k}$, $j \in \Omega$,

Waring's formula

If $X_t^1, ..., X_t^n$ are n dependent Bernoulli r.v. and $\Gamma \subset \Omega$ with cardinal m,

$$\mathrm{P}\left[\sum_{i\in\Gamma} X_t^i = k\right] = \mathbb{1}_{k\leq m} C_m^k \sum_{j=0}^{m-k} C_{m-k}^j (-1)^j \mu_{j+k,\,t}(\Gamma).$$

with

$$\mu_{k,\,t}(\Gamma) = \frac{1}{C^k_{card(\Gamma)}} \sum_{\substack{j_1<j_2<..<j_k \\ j_1,\ldots,j_k\in\Gamma}} \mathrm{P}\left[X_t^{j_1} = 1 \cap ... \cap X_t^{j_k} = 1\right], \quad k \geq 1,$$

$$\mu_{0,\,t}(\Gamma) = 1 \text{ (even if } \Gamma = \emptyset).$$

Appendix I - Probabilistic tools

Interest in life-insurance framework :

- independence assumptions
- but different ages and non identically distributed lifetimes

Interest for Davis and Lo extension :

- one would like $\mathrm{P}\left[N=k\right]$
- on can change more easily iid assumptions
- is simplified with exchangeability assumptions

Appendix I - Probabilistic tools

Idea from so-called Waring's formula

for non iid Bernoulli r.v. $A_1, \dots, A_n$, one can get the law of $\sum_j A_j$ as a function of coefficients of kind

$$\mathrm{P}\left[A_1 = 1 \cap \dots \cap A_i = 1\right].$$

- If independence : these coefficients become products
- If exchangeability : these coefficients does only depend on the number of considered r.v.

Here we are looking for :

- Directs defaults : $\sum_{j \in \Gamma} X_t^j$ as a function of coefficients $\mu_{k,t}(\Gamma)$,
- Contagion links : $\sum_{j \in F_t} Y_t^{\sigma(j)}$ as a function of coefficients $\lambda_{k,t}$,
- Indirects defaults : $\sum_{j=1\dots k} \mathscr{C}_t^j$ as a function of coefficients $\xi_{k,t}$,

Appendix II - Basic numerical illustration

Appendix II - Basic numerical illustration

we consider here that for all t,

- X_t^i are exchangeables, Bernoulli with hidden parameter Θ_X, $\mathrm{E}[\Theta_X] = p = 0.1$, $\mathrm{V}[\Theta_X]$ is given
- Y_t^{ij} are exchangeables, Bernoulli with hidden parameter Θ_Y, $\mathrm{E}[\Theta_Y] = q = 0.2$, $\mathrm{V}[\Theta_Y]$ is given
- hidden parameters are Beta distributed

We consider

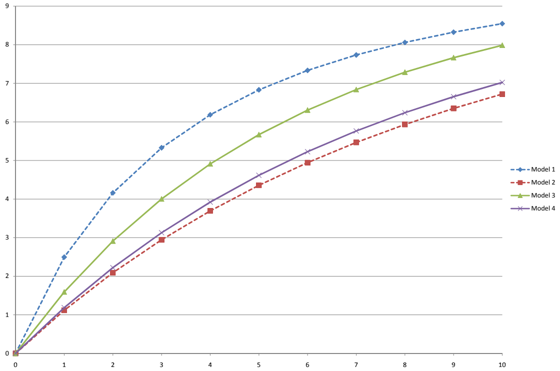
- 10 entities ($n = 10$),
- 10 temporal units ($T = 10$),
- average direct default probability $p = 0.1$,
- average contagion link probability $q = 0.2$.

Appendix II - Basic numerical illustration

We define 4 models with common parameters :

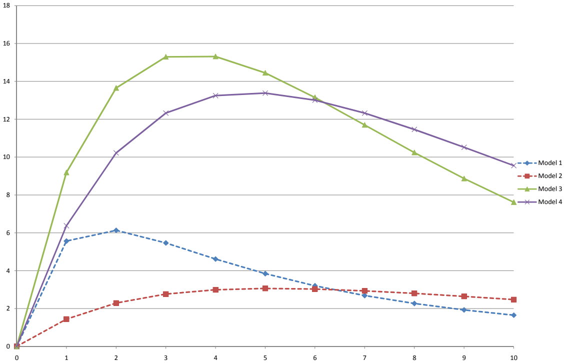
1. *model 1* : $\sigma_X = 0$, $\sigma_Y = 0$, $f(x) = \mathbb{1}_{x \geqslant 1}$ (i.i.d. case, one contagion link required).
2. *model 2* : $\sigma_X = 0$, $\sigma_Y = 0$, $f(x) = \mathbb{1}_{x \geqslant 2}$ (i.i.d. case, two contagion links required).
3. *model 3* : $\sigma_X = 0.2$, $\sigma_Y = 0.2$, $f(x) = \mathbb{1}_{x \geqslant 1}$ (exchangeable case, one contagion link required).
4. *model 4* : $\sigma_X = 0.2$, $\sigma_Y = 0.2$, $f(x) = \mathbb{1}_{x \geqslant 2}$ (exchangeable case, two contagion link required).

Appendix II - Basic numerical illustration

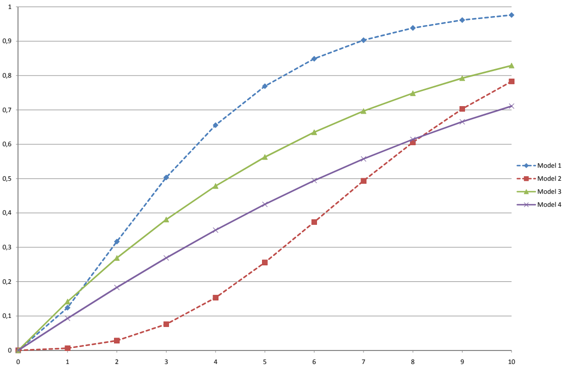


Evolution of $\mathrm{E}[N_t]$ as a function of t. i.i.d. case dotted.

Appendix II - Basic numerical illustration

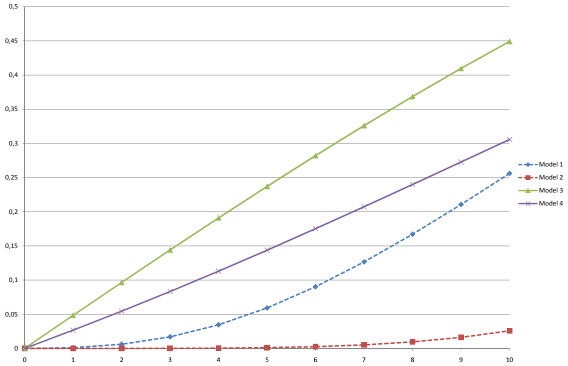


Evolution of $\mathbb{V}[N_t]$ as a function of t. i.i.d. case dotted.



Evolution of $\mathrm{P}\left[N_t \geqslant 6\right]$ as a function of t. i.i.d. case dotted.

Appendix II - Basic numerical illustration



Evolution of $\mathrm{P}[N_t \geqslant 10]$ as a function of t. i.i.d. case dotted.

Appendix III - Some remarks

specificity of the model

- try to capture explicit microstructure of contagion
- contagion acts directly on random variables, not on probabilities
- one can say with certainty if default of entity i is due to entity j
- acts in a complete graph

some limits of the model

- default rate depends on the number n of entities
- contagions only within the considered portofolio
- numerical issues for large number n of entities

some perspectives

- recursions to manage numerical issues
- contagions from outside the portofolio
- behavior when time tends to zero and n becomes large
- asymptotic results - larger interconnected component
- recovery effects
- Heterogeneity via a small number of groups