Pricing and Hedging Loss Derivatives in a Markovian Bottom-Up Model

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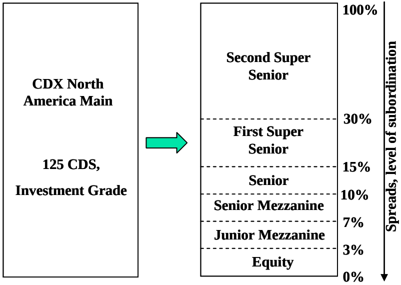
Pricing and Hedging Portfolio Credit Derivatives in a Bottom-up Model with Simultaneous Defaults

Introduction

Risk management of portfolio credit derivatives



- Cash-flows driven by the realized path of the aggregate loss process

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)N_t^i$$

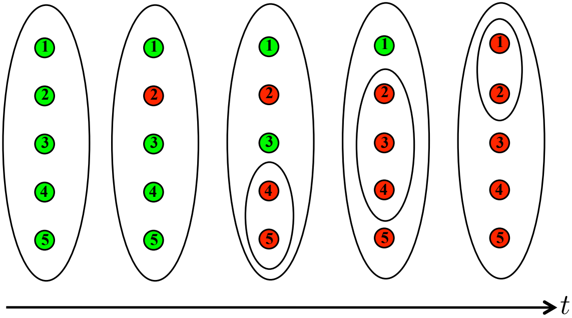
where R_i is the recovery rate and N_t^i is the default indicator of obligor i

Markovian portfolio credit risk model

Simultaneous default model

- Defaults are the consequence of trigger events affecting simultaneously pre-specified groups of obligors

Example: $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$.



Markovian portfolio credit risk model

- $\{1, \ldots, n\}$: credit references
- $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$: pre-specified groups
- $\lambda_Y(\cdot)$ intensity function associated with group Y
- N_t^i default indicator process of name $i = 1, \ldots, n$
- $\mathbf{N}_t = (N_t^1, \ldots, N_t^n)$ is a multivariate Markov chain in $\{0,1\}^n$ such that for $\mathbf{k}, \mathbf{m} \in \{0,1\}^n$:

$$\mathbb{P}\left(\mathbf{N}_{t+dt} = \mathbf{m} \mid \mathbf{N}_t = \mathbf{k}\right) = \sum_{Y \in \mathcal{Y}} \lambda_Y(t) \mathbf{1}_{\{\mathbf{k}^Y = \mathbf{m}\}} dt$$

where $\mathbf{k}^Y$ is obtained from $\mathbf{k} = (k_1, \ldots, k_n)$ by replacing the components k_j, $j \in Y$, by number one.

Pricing and Hedging

Markov copula condition

For any $i = 1, \dots, n$, N^i is a one dimensional Markov process:

$$\mathbb{E}\left[\Phi(N_T^i) \mid N_t^1, \dots, N_t^n\right] = \mathbb{E}\left[\Phi(N_T^i) \mid N_t^i\right]$$

Independent pricing and calibration of single-name products

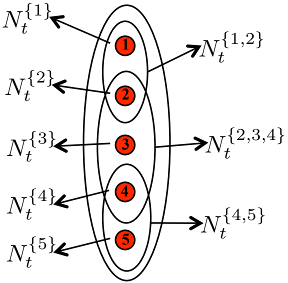
Hedging CDO tranches with single-name CDS

- Dynamics of CDO tranche prices and single-name CDS can be expressed in terms of some fundamental martingales
- Computation of min-variance hedging strategies
- Price of portfolio derivatives solves the Kolmogorov backward equations

Numerically intractable at least for large heterogeneous portfolios $(n > 20)$

Common Shocks Model Interpretation

Example: $n = 5$ and $\mathcal{Y} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{4,5\},\{2,3,4\},\{1,2\}\}$.



$$\hat{N}_t^1 := \max\left\{N_t^{\{1\}}, N_t^{\{1,2\}}\right\}$$
$$\hat{N}_t^2 := \max\left\{N_t^{\{2\}}, N_t^{\{1,2\}}, N_t^{\{2,3,4\}}\right\}$$
$$\hat{N}_t^3 := \max\left\{N_t^{\{3\}}, N_t^{\{2,3,4\}}\right\}$$
$$\hat{N}_t^4 := \max\left\{N_t^{\{4\}}, N_t^{\{2,3,4\}}, N_t^{\{4,5\}}\right\}$$
$$\hat{N}_t^5 := \max\left\{N_t^{\{5\}}, N_t^{\{4,5\}}\right\}$$

- N_t^Y, $Y \in \mathcal{Y}$ are independent $\{0,1\}$–point processes with intensity λ_Y:

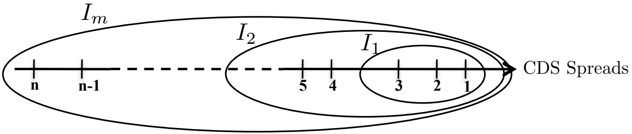
Common Shocks Model Interpretation

$$\left(\hat{N}_{t_1}^1, \ldots, \hat{N}_{t_n}^n\right) \stackrel{d}{=} \left(N_{t_1}^1, \ldots, N_{t_n}^n\right)$$

Common Shocks Model Interpretation

Calibration of individual intensities on single-name CDS

- Individual shocks + Common shocks: $\mathcal{Y} = \{\{1\}, \dots, \{n\}, I_1, \dots, I_m\}$
- Names are ordered with respect to riskiness



- Price of CDS i can be expressed as a function of $\mathbb{E}\left[N_t^i\right]$, $t = 0, \dots, T$

$$\mathbb{E}\left[N_t^i\right] = 1 - \exp\left(-\int_0^t \eta_i(u)dt\right)$$

where

$$\eta_i(u) = \lambda_{\{i\}}(u) + \sum_{k=1}^{m} \lambda_{I_k}(u)\mathbf{1}_{\{i \in I_k\}}$$

- η_i, $i = 1, \dots, n$ calibrated on individual CDS spreads by a bootstrap procedure

Common Shocks Model Interpretation

Calibration of common-shocks intensities on CDO tranches

- Pricing of CDO tranches only involves marginal loss distributions
- Thanks to the common-shock model interpretation:

$$L_t = \frac{1}{n}\sum_{i=1}^{n}(1-R_i)N_t^i \overset{d}{=} \frac{1}{n}\sum_{i=1}^{n}(1-R_i)\hat{N}_t^i$$

- Conditionally on $\left(N_t^{I_1},\dots,N_t^{I_m}\right)$, $\hat{N}^1,\dots,\hat{N}^n$ are independent Bernoulli random variables with parameters

$$p_t^i = \begin{cases} 1 & i \in \cup_{k=1}^{m}\{I_k\,;\,\hat{N}_t^{I_k}=1\} \\ 1-\exp\left(-\int_0^t \lambda_{\{i\}}(u)du\right) & \text{else} \end{cases}$$

where

$$\lambda_{\{i\}}(u) = \eta_i(u) - \sum_{k=1}^{m}\lambda_{I_k}(u)\mathbf{1}_{\{i\in I_k\}} \geq 0$$

Fast convolution-recursion procedure for computing loss distributions

Numerical Results

Data set: 5-years CDX North-America index on 20 December 2007

- Quoted spreads (at different pillars) of the 125 index constituents
- Quoted spreads of standard CDO tranches

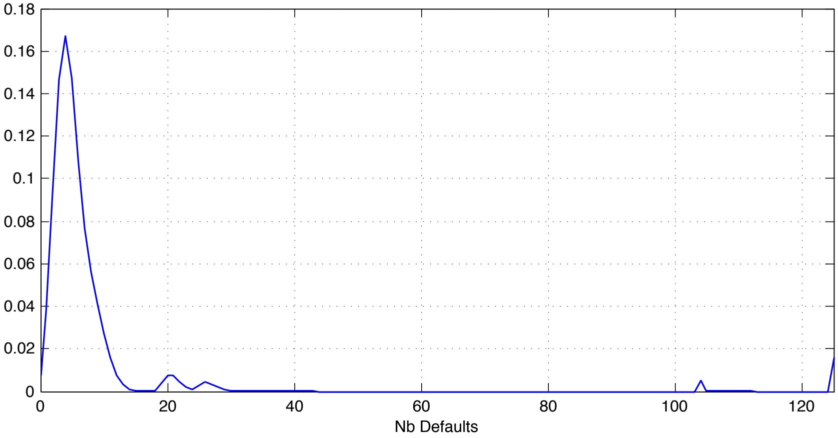
Model specification:

- 5 groups $I_1 \subset \cdots \subset I_5$ such that $I_1 = \{1, \ldots, 6\}$, $I_2 = \{1, \ldots, 18\}$, $I_3 = \{1, \ldots, 25\}$, $I_4 = \{1, \ldots, 100\}$, $I_5 = \{1, \ldots, 125\}$
- Piecewise constant intensities $\lambda_{\{1\}}, \ldots, \lambda_{\{125\}}$, $\lambda_{I_1}, \ldots, \lambda_{I_5}$ with grid points corresponding to CDS pillars
- Recovery rate: 40%
- Interest rate: 3%

Calibration results:

	0%-3%	3%-7%	7%-10%	10%-15%	15%-30%	index
Market quotes	54	265	125	62	43	81
Model outputs	48	254	124	61	41	78

5-years calibrated loss distribution:



Future Research

- Hedging CDO tranches with individual CDS
- Computation of min-variance hedging strategies
- Comparison with Gaussian copula spread-sensitivity deltas

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