Dynamic hedging of synthetic CDO tranches

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Introduction

- In this presentation, we address the hedging issue of CDO tranches in a market model where pricing is connected to the cost of the hedge
- In credit risk market, models that connect pricing to the cost of the hedge have been studied quite lately
- Discrepancies with the interest rate or the equity derivative market
- Model to be presented is not new, require some stringent assumptions, but the hedging can be fully described in a dynamical way

Introduction

Presentation related to the papers :

- *Hedging default risks of CDOs in Markovian contagion models* (2008), to appear in Quantitative Finance, with Jean-Paul Laurent and Jean-David Fermanian
- *Hedging CDO tranches in a Markovian environment* (2009), book chapter with Monique Jeanblanc and Jean-Paul Laurent
- *Hedging portfolio loss derivatives with CDSs* (2010), working paper with Monique Jeanblanc
- *Delta-hedging correlation risk ?* (2010), working paper with Rama Cont, Stéphane Crépey and Yu Hang Kan
- *Dynamic hedging of synthetic CDO tranches : Bridging the gap between theory and practice* (2010), book chapter with Jean-Paul Laurent

Contents

Default times

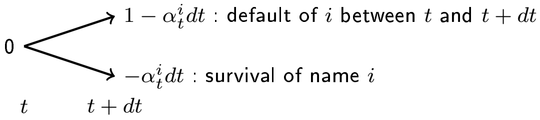
- n credit references
- $\tau_1, \ldots, \tau_n$: default times defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- $N_t^i = 1_{\{\tau_i \leq t\}}$, $i = 1, \ldots, n$: default indicator processes
- $\mathbb{H}^i = (\mathcal{H}_t^i)_{t \geq 0}$, $\mathcal{H}_t^i = \sigma(N_s^i,\ s \leq t)$, $i = 1, \ldots, n$: natural filtration of N^i
- $\mathbb{H} = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$: global filtration of default times

Default times

- No simultaneous defaults : $\mathbb{P}(\tau_i = \tau_j) = 0, \forall i \neq j$
- Default times admit $\mathbb{H}$-adapted default intensities
  - For any $i = 1, \ldots, n$, there exists a non-negative $\mathbb{H}$-adapted process $\alpha^{i,\mathbb{P}}$ such that
    $$M_t^{i,\mathbb{P}} := N_t^i - \int_0^t \alpha_s^{i,\mathbb{P}} ds$$
    is a $(\mathbb{P}, \mathbb{H})$-martingale.
  - $\alpha_t^{i,\mathbb{P}} = 0$ on the set $\{t > \tau_i\}$
  - $M^{i,\mathbb{P}}$, $i = 1, \ldots, n$ will be referred to as the fundamental martingales

Market Assumption

- Instantaneous digital CDS are traded on the names $i = 1, \dots, n$
- Instantaneous digital CDS on name i at time t is a stylized bilateral agreement
  - Offer credit protection on name i over the short period $[t, t + dt]$
  - Buyer of protection receives 1 monetary unit at default of name i
  - In exchange for a fee equal to $\alpha_t^i dt$



- Cash-flow at time $t + dt$ (buy protection position) : $dN_t^i - \alpha_t^i dt$
- $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$ (Contrat is worthless)

Market Assumption

- Credit spreads are driven by defaults : $\alpha^1, \ldots, \alpha^n$ are $\mathbb{H}$-adapted processes
- Payoff of a self-financed strategy

$$V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_s^i e^{r(T-s)} \underbrace{\left(dN_s^i - \alpha_s^i ds\right)}_{\text{CDS cash-flow}}.$$

- r : default-free interest rate
- V_0 : initial investment
- δ^i, $i = 1, \ldots, n$, $\mathbb{H}$-predictable process

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ *be a* $\mathbb{P}$*-integrable random variable. Then, there exists* $\mathbb{H}$*-predictable processes* $\theta^i, i = 1, \ldots, n$ *such that*

$$\begin{aligned} A &= \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i \left(dN_s^i - \alpha_s^{i,\mathbb{P}} ds \right) \\ &= \mathbb{E}_{\mathbb{P}}[A] + \sum_{i=1}^{n} \int_0^T \theta_s^i dM_s^{i,\mathbb{P}} \end{aligned}$$

and $\mathbb{E}_{\mathbb{P}} \left(\int_0^T |\theta_s^i| \alpha_s^{i,\mathbb{P}} ds \right) < \infty$.

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ *be a* $\mathbb{Q}$*-integrable random variable. Then, there exists* $\mathbb{H}$*-predictable processes* $\hat{\theta}^i, i = 1, \ldots, n$ *such that*

$$A = \mathbb{E}_{\mathbb{Q}}[A] + \sum_{i=1}^{n} \int_0^T \hat{\theta}^i_s \underbrace{\left(dN^i_s - \alpha^i_s ds\right)}_{\text{CDS cash-flow}}$$

$$= \mathbb{E}_{\mathbb{Q}}[A] + \sum_{i=1}^{n} \int_0^T \hat{\theta}^i_s dM^i_s$$

and $\mathbb{E}_{\mathbb{Q}}\left(\int_0^T |\theta^i_s| \alpha^{i,\mathbb{P}}_s ds\right) < \infty$.

Hedging and martingale representation theorem

Building a change of probability measure

- Describe what happens to default intensities when the original probability is changed to an equivalent one
- From the PRT, any Radon-Nikodym density ζ (strictly positive $(\mathbb{P},\mathbb{H})$-martingale with expectation equal to 1) can be written as

$$d\zeta_t = \zeta_{t-} \sum_{i=1}^{n} \pi_t^i dM_t^{i,\mathbb{P}}, \ \zeta_0 = 1$$

where π^i, $i = 1, \ldots, n$ are $\mathbb{H}$-predictable processes

Hedging and martingale representation theorem

- Conversely, the (unique) solution of the latter SDE is a local martingale (Doléans-Dade exponential)

$$\zeta_t = \exp\left(-\sum_{i=1}^{n}\int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds\right)\prod_{i=1}^{n}(1+\pi_{\tau_i}^i)^{N_t^i}$$

- The process ζ is non-negative if $\pi^i > -1$, for $i = 1, \ldots, n$
- The process ζ is a true martingale if $\mathbb{E}_{\mathbb{P}}\left[\zeta_t\right] = 1$ for any t or if π^i is bounded, for $i = 1, \ldots, n$

Hedging and martingale representation theorem

Theorem (Change of probability measure)

Define the probability measure $\mathbb{Q}$ *as*

$$d\mathbb{Q}|_{\mathcal{H}_t} = \zeta_t d\mathbb{P}|_{\mathcal{H}_t} \, .$$

where

$$\zeta_t = \exp\left(-\sum_{i=1}^{n} \int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds \right) \prod_{i=1}^{n} (1 + \pi_{\tau_i}^i)^{N_t^i}$$

Then, for any $i = 1, \dots, n$, *the process*

$$M_t^i := M_t^{i,\mathbb{P}} - \int_0^t \pi_s^i \alpha_s^{i,\mathbb{P}} ds = N_t^i - \int_0^t (1 + \pi_s^i) \alpha_s^{i,\mathbb{P}} ds$$

is a $\mathbb{Q}$*-martingale. In particular, the* $(\mathbb{Q}, \mathbb{H})$*-intensity of* τ_i *is* $(1 + \pi_t^i)\alpha_t^{i,\mathbb{P}}$.

Hedging and martingale representation theorem

- From the absence of arbitrage opportunity

$$\left\{\alpha_t^i > 0\right\} \overset{\mathbb{P}-a.s.}{=} \left\{\alpha_t^{i,\mathbb{P}} > 0\right\}$$

- For any $i = 1, \dots, n$, the process $\hat{\pi}^i$ defined by :

$$\hat{\pi}_t^i = \left(\frac{\alpha_t^i}{\alpha_t^{i,\mathbb{P}}} - 1\right)\left(1 - N_{t-}^i\right)$$

  is an $\mathbb{H}$-predictable process such that $\hat{\pi}^i > -1$
- The process ζ defined with $\pi^1 = \hat{\pi}^1, \dots, \pi^n = \hat{\pi}^n$ is an admissible Radon-Nikodym density
- Under $\mathbb{Q}$, credit spreads $\alpha^1, \dots, \alpha^n$ are exactly the intensities of default times

Hedging and martingale representation theorem

- The predictable representation theorem also holds under $\mathbb{Q}$
- In particular, if A is an $\mathcal{H}_T$ measurable payoff, then there exists $\mathbb{H}$-predictable processes $\hat{\theta}^i, i = 1, \dots, n$ such that

$$A = \mathbb{E}_{\mathbb{Q}}\left[A \mid \mathcal{H}_t\right] + \sum_{i=1}^{n} \int_t^T \hat{\theta}_s^i \underbrace{\left(dN_s^i - \alpha_s^i ds\right)}_{\text{CDS cash-flow}}.$$

- Starting from t the claim A can be replicated using the self-financed strategy with
  - the initial investment $V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} A \mid \mathcal{H}_t\right]$ in the savings account
  - the holding of $\delta_s^i = \hat{\theta}_s^i e^{-r(T-s)}$ for $t \le s \le T$ and $i = 1, \dots, n$ in the instantaneous CDS
- As there is no charge to enter a CDS, the replication price of A at time t is $V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} A \mid \mathcal{H}_t\right]$

Hedging and martingale representation theorem

- A depends on the default indicators of the names up to time T
  - includes the cash-flows of CDO tranches or basket credit default swaps, given deterministic recovery rates
- The latter theoretical framework can be extended to the case where actually traded CDS are considered as hedging instruments
  - See Cousin and Jeanblanc (2010) for an example with a portfolio composed of 2 names or in a general n-dimensional setting when default times are assumed to be ordered

Hedging and martingale representation theorem

- Risk-neutral measure can be explicitly constructed
  - We exhibit a continuous change of probability measure
- Completeness of the credit market stems from a martingale representation theorem
  - Perfect replication of claims which depend only upon the default history with CDS on underlying names and default-free asset
  - Provide the replication price at time t
- But does not provide any operational way of constructing hedging strategies
- Markovian assumption is required to effectively compute hedging strategies

Markovian contagion model

- Pre-default intensities only depend on the **current status of defaults**

$$\alpha_t^i = \tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right) 1_{t<\tau_i}, \; i = 1, \ldots, n$$

- Ex : Herbertsson - Rootzén (2006)

$$\tilde{\alpha}^i \left(t, N_t^1, \ldots, N_t^n\right) = a_i + \sum_{j \neq i} b_{i,j} N_t^j$$

- Ex : Lopatin (2008)

$$\tilde{\alpha}^i \left(t, N_t\right) = a_i(t) + b_i(t) f(t, N_t)$$

- Connection with continuous-time Markov chains
  - $\left(N_t^1, \ldots, N_t^n\right)$ Markov chain with possibly 2^n states
  - Default times follow a multivariate phase-type distribution

Homogeneous Markovian contagion model

- Pre-default intensities only depend on the current number of defaults
- All names have the same pre-default intensities

$$\alpha_t^i = \tilde{\alpha}(t, N_t) 1_{t<\tau_i}, \ i = 1, \dots, n$$

where

$$N_t = \sum_{i=1}^{n} N_t^i$$

- This model is also referred to as the local intensity model

Homogeneous Markovian contagion model

- No simultaneous default, the intensity of N_t is equal to

$$\lambda(t, N_t) = (n - N_t)\tilde{\alpha}(t, N_t)$$

- N_t is a continuous-time Markov chain (pure birth process) with generator matrix :

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Model involves as many parameters as the number of names

Homogeneous Markovian contagion model

Replication price of a European type payoff

$$V(t,k) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(N_T) \mid N_t = k\right]$$

- $V(t,k)$, $k = 0,\ldots,n-1$ solve the backward Kolmogorov differential equations :

$$\frac{\delta V(t,k)}{\delta t} = rV(t,k) - \lambda(t,k)\left(V(t,k+1) - V(t,k)\right)$$

- Approach also puts in practice by van der Voort (2006), Schönbucher (2006), Herbersson (2007), Arnsdorf and Halperin (2007), Lopatin and Misirpashaev (2007), Cont and Minca (2008), Cont and Kan (2008), Cont, Deguest and Kan (2009)

Homogeneous Markovian contagion model

Computation of credit deltas

- $V(t, N_t)$, price of a CDO tranche (European type payoff)
- $V^I(t, N_t)$, price of the CDS index (European type payoff)

$$V(t, N_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(N_T) \mid N_t\right]$$

$$V^I(t, N_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\Phi^I(N_T) \mid N_t\right]$$

- Using standard Itô's calculus

$$dV\left(t, N_t\right) = \left(V\left(t, N_t\right) - \delta^I(t, N_t)V^I\left(t, N_t\right)\right) rdt + \delta^I(t, N_t)dV^I\left(t, N_t\right)$$

where

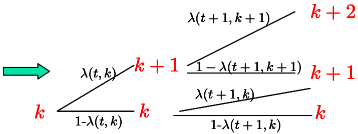
$$\delta^I(t, N_t) = \frac{V\left(t, N_t + 1\right) - V\left(t, N_t\right)}{V^I\left(t, N_t + 1\right) - V^I\left(t, N_t\right)}.$$

- Perfect replication with the index and the risk-free asset

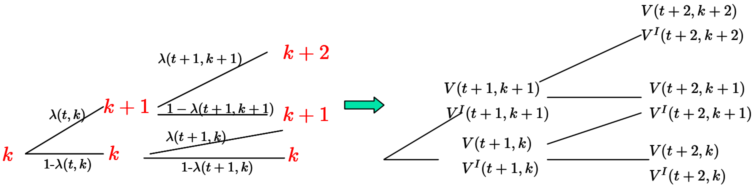
Pricing and hedging in a binomial tree

- Binomial tree : discrete version of the homogeneous contagion model

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

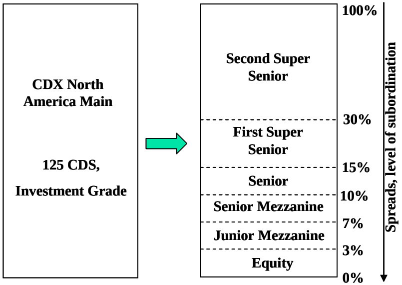


- Given some loss intensities $\lambda(t,k)$, CDO tranches and index price computed by backward induction :



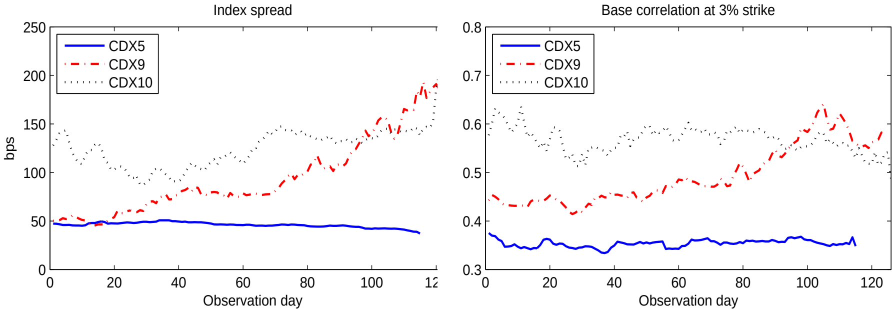
Empirical results

- Slice the credit portfolio into different risk levels or CDO tranches
- ex : CDO tranche on standardized Index such as CDX North America Investment Grade



Empirical results

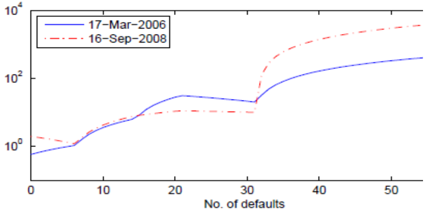
- 5-year CDX NA IG Series 5 from 20 September 2005 to 20 March 2006
- 5-year CDX NA IG Series 9 from 20 September 2007 to 20 March 2008
- 5-year CDX NA IG Series 10 from 21 March 2008 to 20 September 2008



Empirical results

Two different calibration methods used to fit loss intensities

- **Para**metric method : $\lambda(t,k) = \lambda(k) = (n-k)\sum_{i=0}^{k} b_i$ (Herbertsson (2008))
- **E**ntropy **M**inimisation algorithm calibration : $\inf_{\mathbb{Q}\in\Lambda} \mathbb{E}^{\mathbb{Q}_0}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{Q}_0}\right)\right]$ subject to the calibration constraints (Cont and Minca (2008))



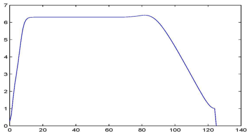


Figure 2: Dependence of default intensity on number of defaults for $t = 1$ year: ITRAXX Europe Series 6, March 15 2007..

- Left : Cont, Cousin, Crépey and Kan (2010), Right : Cont and Minca (2008)

Empirical results

Details of calibration results – 5-year CDX.NA.IG Series 9 on 20 September 2007 (bps excepted for the equity tranche quoted in percentage

Tranche	Market	Gauss	Para	EM
Index	50.38	50.36	47.58	47.58
0%-3%	35.55	35.55	36.35	36.35
3%-7%	131.44	131.44	132.04	132.07
7%-10%	45.51	45.51	45.54	45.56
10%-15%	25.28	25.28	25.30	25.31
15%-30%	15.24	15.24	15.36	15.36

Empirical results

Comparison of three alternative hedging methods

- Gauss delta : index Spread sensitivity computed in a one-factor Gaussian copula model

$$\Delta_t^{\text{Gauss}} = \frac{\mathcal{V}(t, S_t + \varepsilon, \rho_t) - \mathcal{V}(t, S_t, \rho_t)}{\mathcal{V}^I(t, S_t + \varepsilon) - \mathcal{V}^I(t, S_t)}$$

  where $\mathcal{V}$ and $\mathcal{V}^I$ are the Gaussian copula pricing function associated with (resp.) the tranche and the CDS index.

- Local intensity delta :

$$\delta^I(t, N_t) = \frac{V(t, N_t + 1) - V(t, N_t)}{V^I(t, N_t + 1) - V^I(t, N_t)}.$$

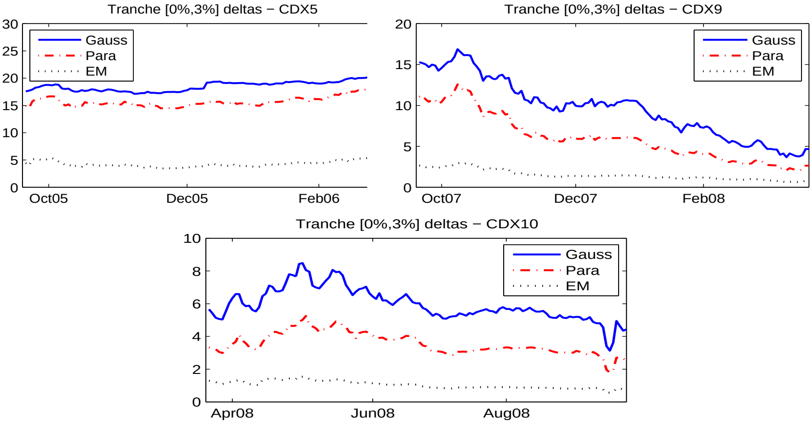
  with both Parametric (Param) and Entropy Minimisation (EM) calibration methods

Credit deltas on 20 September 2007 (normalized to tranche notional)

Tranche	Gauss	Para	EM
0%-3%	15.29	11.05	2.64
3%-7%	5.03	4.59	2.70
7%-10%	1.94	2.26	2.29
10%-15%	1.10	1.47	1.99
15%-30%	0.60	1.01	1.74

Empirical results

Dynamics of $[0\%, 3\%]$-equity tranche credit deltas, CDX.NA.IG series 5, 9 and 10



Hedging performance for 1-day rebalancing

Back-testing hedging experiments on series 5, 9 and 10 (1-day rebalancing)

$$\text{Relative hedging error} = \left|\frac{\text{Average P\&L of the hedged position}}{\text{Average P\&L of the unhedged position}}\right|,$$

$$\text{Residual volatility} = \frac{\text{P\&L volatility of the hedged position}}{\text{P\&L volatility of the unhedged position}}.$$

Relative hedging errors (in percentage) :

	CDX5			CDX9			CDX10		
Tranche	Li	Para	EM	Li	Para	EM	Li	Para	EM
0%-3%	4.01	5.48	39.57	80.70	10.77	59.20	33.01	55.59	88.89
3%-7%	1.25	3.29	9.66	0.42	19.89	51.67	48.09	49.64	77.25
7%-10%	10.65	10.42	117.14	15.70	13.71	29.36	49.63	25.19	41.58
10%-15%	7.22	27.08	229.00	27.78	18.73	11.06	139.82	181.78	214.14
15%-30%	0.54	61.19	355.26	3.66	32.66	88.86	172.78	269.83	452.67

Hedging performance for 1-day rebalancing

Residual volatilities (in percentage) :

	CDX5			CDX9			CDX10		
Tranche	Gauss	Para	EM	Gauss	Para	EM	Gauss	Para	EM
0%-3%	45.85	47.70	66.80	59.62	59.70	79.03	105.01	91.06	89.50
3%-7%	70.76	72.25	77.54	58.20	47.54	55.21	85.02	74.40	72.97
7%-10%	90.86	101.72	164.36	53.19	50.88	44.55	83.30	79.67	69.74
10%-15%	90.52	107.63	254.57	61.01	63.20	62.57	91.83	93.81	89.26
15%-30%	93.86	110.95	271.44	37.42	49.02	73.01	84.39	99.97	131.12

Conclusion :

- Hedging based on local intensity model with Entropy Minimisation calibration gives poor performance
- No clear evidence to distinguish the performance based on the Gaussian copula model and the parametric local intensity model