




On Multivariate Extensions of Value-at-Risk


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Joint work with Elena Di Bernardino, CNAM

Workshop "Des risques factoriels aux mesures de risque systémiques"

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-  A. Cousin, E. Di Bernardino, *On Multivariate Extensions of Value-at-Risk*, submitted to *Journal of Multivariate Analysis*
Available on HAL: <http://hal.archives-ouvertes.fr/hal-00638382>

-  A. Cousin, E. Di Bernardino, *On Multivariate Extensions of Conditional-Tail-Expectation*, in preparation

- Regulatory capital rule relies on the VaR paradigm and the risk diversification effect.
- How could we deal with risks that cannot be aggregated together ?
- Presence of non-monetary risks ?
- Exogenous risks ?

Construction of Multivariate Risk Measures

$$\rho: \mathbf{X} := (X_1, \dots, X_d) \mapsto \begin{pmatrix} \rho^1[\mathbf{X}] \\ \vdots \\ \rho^d[\mathbf{X}] \end{pmatrix} \in \mathbb{R}_+^d,$$

Some desirable properties:

- Combine in a concise way information on both marginals and risks dependencies
- Compatible with univariate version when $d = 1$
- Easily computable for large class of multivariate distribution functions
- Consistent with usual invariance properties (Artzner et al.'s axioms)
- Consistent behavior with respect to risk perturbations

Multivariate *Value-at-Risk* as quantile curve (Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009):

$$\partial \underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) = \alpha\}$$

$$\partial \bar{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : \bar{F}(\mathbf{x}) = 1 - \alpha\}$$

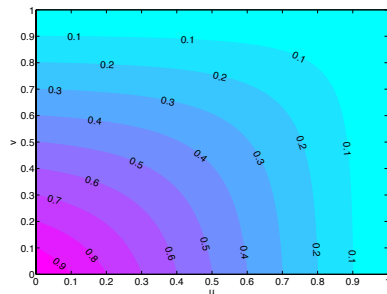
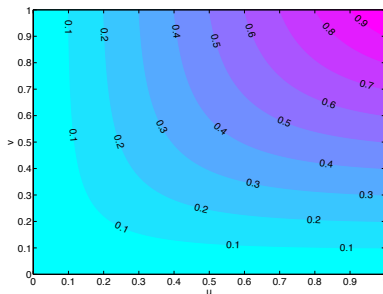


Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Value-at-Risk**

Definition

Consider a random vector \mathbf{X} with absolutely continuous cdf F and survival function \bar{F} . For $\alpha \in (0, 1)$, we define:

$$\underline{\text{VaR}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) = \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | F(\mathbf{X}) = \alpha] \\ \vdots \\ \mathbb{E}[X_d | F(\mathbf{X}) = \alpha] \end{pmatrix}$$

$$\overline{\text{VaR}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \bar{F}(\mathbf{X}) = 1 - \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | \bar{F}(\mathbf{X}) = 1 - \alpha] \\ \vdots \\ \mathbb{E}[X_d | \bar{F}(\mathbf{X}) = 1 - \alpha] \end{pmatrix}$$

When $d = 1$: $\underline{\text{VaR}}_{\alpha}(X) = \overline{\text{VaR}}_{\alpha}(X) = \text{VaR}_{\alpha}(X)$

VaRs for risk portfolios with Archimedean copula dependence structure

Proposition

Let \mathbf{X} be a d -dimensional portfolio of risks with marginal distributions F_1, \dots, F_d .

- If \mathbf{X} admits an *Archimedean copula* with generator ϕ , then

$$\underline{\text{VaR}}_{\alpha}^i(\mathbf{X}) = \mathbb{E} \left[F_i^{-1} \left(\phi^{-1}(S_i \phi(\alpha)) \right) \right], \quad i = 1, \dots, d$$

- If $\tilde{\mathbf{X}}$ admits an *Archimedean survival copula* with generator ϕ , then

$$\overline{\text{VaR}}_{\alpha}^i(\tilde{\mathbf{X}}) = \mathbb{E} \left[F_i^{-1} \left(1 - \phi^{-1}(S_i \phi(1 - \alpha)) \right) \right], \quad i = 1, \dots, d$$

where S_i is a random variable with $\text{Beta}(1, d - 1)$ distribution.

Explicit expressions for bivariate Clayton copulas

Copula	θ	$\underline{\text{VaR}}_{\alpha, \theta}^i(X, Y)$	$\overline{\text{VaR}}_{\alpha, \theta}^i(\tilde{X}, \tilde{Y})$
Clayton C_θ	$(-1, \infty)$	$\frac{\theta}{\theta-1} \frac{\alpha^\theta - \alpha}{\alpha^{\theta-1}}$	$1 - \frac{\theta}{\theta-1} \frac{(1-\alpha)^\theta - (1-\alpha)}{(1-\alpha)^{\theta-1}}$
Counter-monotonic	-1	$\frac{1+\alpha}{2}$	$\frac{\alpha}{2}$
Independent	0	$\frac{\alpha-1}{\ln \alpha}$	$1 + \frac{\alpha}{\ln(1-\alpha)}$
Comonotonic	∞	α	α

Table: Components $i = 1, 2$ of $\underline{\text{VaR}}^i$ and $\overline{\text{VaR}}^i$ where (X, Y) follows a Clayton copula and $(\tilde{X}, \tilde{Y}) := (1 - X, 1 - Y)$, i.e., (\tilde{X}, \tilde{Y}) has a survival Clayton copula with uniform margins

Behavior of VaR components: bivariate Clayton case

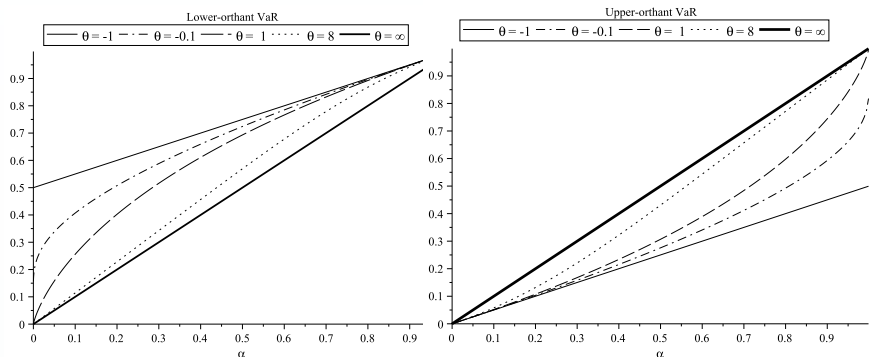


Figure: Behavior of $\underline{\text{VaR}}_{\alpha,\theta}^1(X, Y)$ (left) and $\overline{\text{VaR}}_{\alpha,\theta}^1(\tilde{X}, \tilde{Y})$ (right) with respect to risk level α and dependence parameter θ

Invariance properties, comparison with univariate VaR, behavior with respect to α

	$\underline{\text{VaR}}_\alpha(\mathbf{X})$	$\overline{\text{VaR}}_\alpha(\mathbf{X})$
Several (axiomatic) properties	<p>Invariance properties ($\mathbf{c} \in \mathbb{R}_+^d$):</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha(\mathbf{c}\mathbf{X}) = \mathbf{c} \underline{\text{VaR}}_\alpha(\mathbf{X})$, • $\underline{\text{VaR}}_\alpha(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \underline{\text{VaR}}_\alpha(\mathbf{X})$. <p>Univariate VaR is a lower bound:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) \geq \text{VaR}_\alpha(X_j), \forall \alpha \in (0, 1)$. <p>Comonotonic case:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) = \text{VaR}_\alpha(X_j), \forall \alpha \in (0, 1)$. 	<p>Invariance properties ($\mathbf{c} \in \mathbb{R}_+^d$):</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha(\mathbf{c}\mathbf{X}) = \mathbf{c} \overline{\text{VaR}}_\alpha(\mathbf{X})$, • $\overline{\text{VaR}}_\alpha(\mathbf{c} + \mathbf{X}) = \mathbf{c} + \overline{\text{VaR}}_\alpha(\mathbf{X})$. <p>Univariate VaR is an upper bound:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \text{VaR}_\alpha(X_j), \forall \alpha \in (0, 1)$. <p>Comonotonic case:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) = \text{VaR}_\alpha(X_j), \forall \alpha \in (0, 1)$.
Risk level	<p>$\underline{\text{VaR}}_\alpha^i(\mathbf{X})$ is a non-decreasing function of α. (\mathbf{X} with Archimedean copulas)</p>	<p>$\overline{\text{VaR}}_\alpha^i(\tilde{\mathbf{X}})$ is a non-decreasing function of α. ($\tilde{\mathbf{X}}$ with Archimedean survival copulas)</p>

Effect of a risk perturbation

	$\underline{\text{VaR}}_\alpha(\mathbf{X})$	$\overline{\text{VaR}}_\alpha(\mathbf{X})$
Change in marginals	<p>For a fixed copula C, if $X_i \stackrel{d}{=} Y_i$:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) = \underline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>For a fixed copula C, if $X_i \leq_{st} Y_i$:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \underline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. 	<p>For a fixed copula C, if $X_i \stackrel{d}{=} Y_i$:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) = \overline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>For a fixed copula C, if $X_i \leq_{st} Y_i$:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \overline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$.
Change in dependence structure	<p>For fixed marginals, if $\theta \leq \theta^*$:</p> <ul style="list-style-type: none"> • $\underline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \underline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>$\mathbf{X}$ with Archimedean copula</p>	<p>For fixed marginals, if $\theta \leq \theta^*$:</p> <ul style="list-style-type: none"> • $\overline{\text{VaR}}_\alpha^i(\mathbf{X}) \leq \overline{\text{VaR}}_\alpha^i(\mathbf{Y}), \forall \alpha \in (0, 1)$. <p>$\tilde{\mathbf{X}}$ with Archimedean survival copula</p>

Perspectives

- Are efficient computation procedure available for other kind of dependence structures (heterogeneous one in particular) ?
- What about Multivariate CTE ?
- Comparisons of our multivariate VaR and CTE with existing multivariate risk measures
- Extension to discrete distribution functions

Thank you for your attention

Other multivariate risk measures in the literature

Several multivariate generalizations of CTE. For $i = 1, \dots, d$

- $\text{CTE}_\alpha^{\text{sum}}(X_i) = \mathbb{E}[X_i | S > Q_S(\alpha)]$ where $S = X_1 + \dots + X_d$
- $\text{CTE}_\alpha^{\text{min}}(X_i) = \mathbb{E}[X_i | X_{(1)} > Q_{X_{(1)}}(\alpha)]$ where $X_{(1)} = \min\{X_1, \dots, X_d\}$
- $\text{CTE}_\alpha^{\text{max}}(X_i) = \mathbb{E}[X_i | X_{(d)} > Q_{X_{(d)}}(\alpha)]$ where $X_{(d)} = \max\{X_1, \dots, X_d\}$

For Farlie-Gumbel-Morgenstern copula (Bargès *et al.*, 2009). For elliptic distribution functions (Landsman and Valdez, 2003). For phase-type distributions (Cai and Li, 2005).

- Inappropriate to measure risks with heterogeneous characteristics especially in an external risks problem

Multivariate CTE -s based on upper-level set of multivariate cdf and lower-level set of survival functions:

$$\underline{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : F(\mathbf{x}) \geq \alpha\}$$

$$\bar{L}(\alpha) = \{\mathbf{x} \in \mathbb{R}_+^d : \bar{F}(\mathbf{x}) \leq 1 - \alpha\}$$

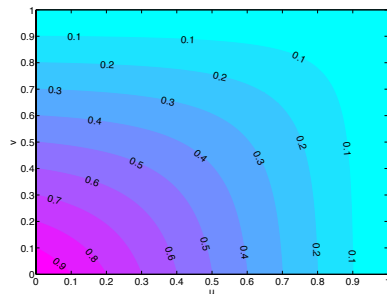
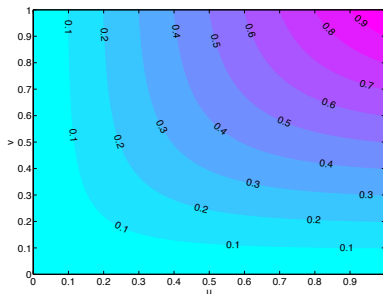


Figure: **left:** quantile curves of Frank copula with parameter 4; **right:** quantile curves of the associated survival distribution function

Lower-Orthant and Upper-Orthant **Conditional-Tail-Expectation**

Definition

Consider a random vector \mathbf{X} with absolutely continuous cdf F and survival function \bar{F} . For $\alpha \in (0, 1)$, we define:

$$\underline{\text{CTE}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | F(\mathbf{X}) \geq \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | F(\mathbf{X}) \geq \alpha] \\ \vdots \\ \mathbb{E}[X_d | F(\mathbf{X}) \geq \alpha] \end{pmatrix}$$

$$\overline{\text{CTE}}_{\alpha}(\mathbf{X}) := \mathbb{E}[\mathbf{X} | \bar{F}(\mathbf{X}) \leq 1 - \alpha] = \begin{pmatrix} \mathbb{E}[X_1 | \bar{F}(\mathbf{X}) \leq 1 - \alpha] \\ \vdots \\ \mathbb{E}[X_d | \bar{F}(\mathbf{X}) \leq 1 - \alpha] \end{pmatrix}$$

CTE explicit expressions for bivariate Clayton copulas

Copula	θ	$\underline{\text{CTE}}_{\alpha, \theta}^i(X, Y)$
Clayton C_θ	$(-1, \infty)$	$\frac{1}{2} \frac{\theta}{\theta-1} \frac{\theta-1-\alpha^2(1+\theta)+2\alpha^{1+\theta}}{\theta-\alpha(1+\theta)+\alpha^{1+\theta}}$
Counter-monotonic W	-1	$\frac{1}{4} \frac{1-\alpha^2+2 \ln \alpha}{1-\alpha+\ln \alpha}$
Independent Π	0	$\frac{1}{2} \frac{(1-\alpha)^2}{1-\alpha+\alpha \ln \alpha}$
Comonotonic M	∞	$\frac{1+\alpha}{2}$

Table: Components $i = 1, 2$ of $\underline{\text{CTE}}^i$ for different copula dependence structures.

Interestingly, one can readily show that $\frac{\partial \text{CTE}_{\alpha, \theta}^i}{\partial \alpha} \geq 0$ and $\frac{\partial \text{CTE}_{\alpha, \theta}^i}{\partial \theta} \leq 0$, for $\theta \geq -1$ and $\alpha \in (0, 1)$.

Behavior of CTE components: bivariate Frank case

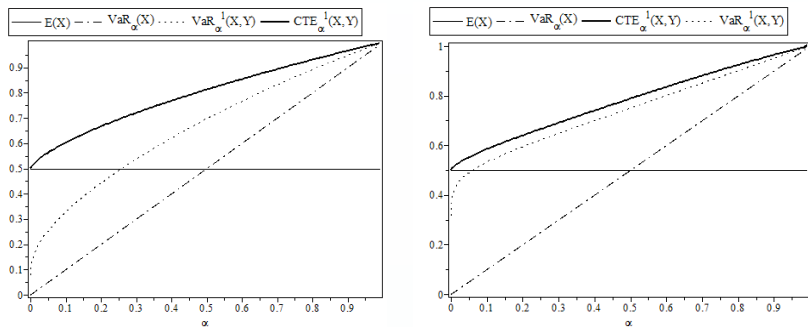


Figure: Frank copula with standard uniform marginals, parameter $\theta = 2$ (left), parameter $\theta = -10$ (right).